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ROBUSTNESS ANALYSIS FOR EVOLUTION SYSTEMS IN HILBERT SPACE: A PASSIVITY APPROACH

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Robustness Analysis for Evolution Systems in Hilbert Space: A Passivity Approach

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Abstract

This report summarizes some recent research results in robustness stability analysis for evolution systems in Hilbert space. With applications such as the control of flexible structures in mind, a general framework is chosen to include infinite dimensional systems and nonlinear, time-varying perturbations. The main result of the report characterizes model perturbations that do not destabilize a nominal closed loop system in terms of the passivity of the nominal system. Special cases of this result produces the generalization of the absolute stability theorem, the hyperstability theorem and the circle criterion to evolution systems. When the perturbation is known to be linear and diagonal, different stability bounds are obtained depending on the signs of the perturbation elements. The directionality in the robustness margins provide possibility to adjust the nominal point of operation to enhance robustness. Robustness of nonlinear nominal systems can also be analyzed by considering the nonlinear dynamics as perturbations. The synthesis problem associated with the passivity approach is shown to be identical to the H_∞-optimization problem. Based on the known solution to the H_∞-optimization problem, we show a procedure for designing stabilizing finite dimensional compensator for infinite dimensional systems. Several examples have been included to illustrate applications of the theoretical results in this report.

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1. Introduction

This report addresses the following problem, defined as the robustness analysis problem, for the interconnected system shown in Fig. 1:

Given a stable system $\mathcal{T}$, find a class of feedback systems, Δ, such that $\mathcal{T}$ remains stable

The situation we are considering is an openloop system with an external feedback compensation to ensure satisfactory performance in the face of system and environmental uncertainties. The design of the feedback compensation is usually based on an approximate model of the open loop system. If the modeling error is treated as a feedback perturbation of the open loop plant, then we have the interconnection as in Fig. 1, where the forward system, $\mathcal{T}$, represents the nominal closed loop system (the *model* of the open loop plant with the feedback compensation) and the feedback system, Δ, represents the modeling error. Possible sources of modeling errors include unmodeled plant dynamics, instrument (sensors and actuators) dynamics, parameter uncertainty etc.

Assume that $\mathcal{T}$ is modeled by an abstract evolution system on a real Hilbert space $\mathbf{X}$:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \quad , \quad x(0) = x_o \in \mathbf{X} \\ y(t) &= Cx(t) + Du(t) \quad . \end{aligned} \tag{1.1}$$

The operator $A : \mathcal{D}(A) \subset \mathbf{X} \to \mathbf{X}$, is the infinitesimal generator of a C_o–semigroup , $U(t)$. The operators, $B : \mathbf{R}^m \to \mathbf{X}$, $C : \mathbf{X} \to \mathbf{R}^m$, $D : \mathbf{R}^m \to \mathbf{R}^m$ are all bounded.

The solution of (1.1) , $x(t)$, is considered in the mild sense [1]:

$$x(t) = U(t)x_o + \int_0^t U(t-s)Bu(s)\,ds \quad . \tag{1.2}$$

The strong differentiability of $x(t)$ is not imposed. Existence and uniqueness of $x(t)$ will be considered later.

$\mathcal{T}$ is said to be exponentially stable if A generates an exponentially stable C_o–semigroup . $x(t) \to 0$ as $t \to \infty$ means that for all $x_o \in \mathbf{X}$, the state trajectory converges to zero in norm. $x(t) \to 0$ exponentially as $t \to \infty$ means that for all $x_o \in \mathbf{X}$, there exists $M(x_o) \in \mathbf{R}$ such that the state trajectory satisfies the bound

$$\|x(t)\| \leq M(x_o)e^{-\sigma t} \quad .$$

In our robustness analysis problem, we assume $\mathcal{T}$ to be exponentially stable. We want to find a class of Δ such that when Δ is connected to $\mathcal{T}$ as a feedback, $x(t) \to 0$ as $t \to \infty$.

The general framework of the abstract evolution system is chosen for the following reasons:

1. Many physical processes are naturally modeled as distributed parameter systems (DPS) that can be cast in the form of (1.1) . Large space structures [2], nuclear reactor dynamics [3, 4], process control systems [5], time delay systems [6], are some examples.

2. This framework provides a global context to study convergence issues related to finite–dimensional approximation models and finite–dimensional compensators based on such models. It has been shown that weak convergence (i.e., the rate of convergence, or, equivalently, the accuracy of the approximation, is dependent on the state) of the gain can occur if proper care has not been taken in the approximation

procedure [7] resulting in high spatial frequency component and possible numerical difficulty in implementation. Loss of controllability and observability has also been noted if an ill-advised basis has been selected for approximation [8].

3. In many applications, this framework provides a more efficient and physically meaningful parameterization of the underlying system. Take a simple Euler-Bernoulli beam for example. The partial differential equation (PDE) model contains only a few physical constants while a high order finite element model incurs a great many more parameters. The robustness analysis problem is also more meaningful if posed with respect to the physical parameters rather than their projections onto some approximation basis.

There have been many robustness analysis techniques proposed in the literature. They can be classified as either frequency domain or time domain methods. In the former category, there are many classical techniques for single-input/single-output (SISO) systems by using, for example, the magnitude and phase plots (Bode plot), Nyquist plot, Nicole's chart etc. For multi-input/multi-output (MIMO) systems, most of the methods are based solely on the gain information, for example, maximum singular values [9] (also known as the principle gain [10] and H_∞-norm [11]) and μ measure [12]. For complex, norm-bounded uncertainties, these criteria are non-conservative. Stability conditions incorporating the phase information of $\mathcal{T}$ have been stated in [10,13] but they do not translate to easily applicable rules. The time domain methods are mostly based on Lyapunov analysis [14,15,16] or Kharitonov's Theorem (see [17] for an introduction). The former studies the solvability of the Lyapunov equation under a perturbed system matrix. The latter deals with the stability of polynomials with uncertain coefficients.

Most of these tools are rooted in finite dimensions and do not apply directly to our general setting. We therefore propose a new robustness analysis technique that is applicable to evolution systems and bridges both time and frequency domains. The stability analysis is performed in the state space by using the Lyapunov method, but the robustness margin is given by an index that is most conveniently computed from the transfer function. This allows us to prove state space stability by working only with a finite dimensional transfer matrix. The main idea of our approach can be stated simply:

Characterize an acceptable class of Δ based on the degree of passivity of $\mathcal{T}$.

The motivation of the passivity approach is based on the following observations:

1. Flexible structures with colocated sensors and acturators are passive. They remain stable for any negative feedback.

2. Passivity analysis is a cornerstone in the field of adaptive control. It is the passivity of certain closed loop transfer function that provides the robustness with respect to the uncertainties in the parameters.

3. Passive systems provide a natural, energy-like Lyapunov function for stability analysis.

4. In the Lur'e problem, passivity is used to characterize systems that remain stable under sector bounded feedback perturbations.

5. Passivity has been used in applications related to the control flexible structures. In [18,19], robust controllers are designed to exploit the open loop passivity property (though the plant needs to be open loop stable). In [20,21], simple adaptive control strategies are devised based on the passivity principle.

6. In the linear quadratic regulator problem (optimal quadratic regulator with full state information), it is known that the closed loop system possesses $[\frac{1}{2},\infty)$ gain margin and $[-\frac{\pi}{3},\frac{\pi}{3}]$ phase margin [22,23]. This fact is most readily seen by noting certain transfer function is positive real (see section 10.1).

Our approach in applying the passivity concept to the robust stability analysis problem is to first study two simpler, prototype problems, the absolute stability and hyperstability problems, and then use to results for the more general situation. A system $\mathcal{T}$ described by (1.1) is called *absolute stable* if $x(t) \to 0$ exponentially as $t \to \infty$ for any feedback system, Δ, that is memoryless, nonlinear time-varying and non-negative. $\mathcal{T}$ is called *hyperstable* if $x(t) \to 0$ as $t \to \infty$ for any dissipative dynamical system, Δ (i.e., Δ satisfies the Popov inequality; see section 2.2). $\mathcal{T}$ is called *exponentially hyperstable* if $x(t) \to 0$ exponentially for any Δ that satisfies the exponential Popov inequality (see section 2.2). The absolute stability (resp. hyperstability) problem is to find a class of $\mathcal{T}$ that is absolute stable (resp. hyperstable).

If $\mathcal{T}$ is finite-dimensional, it was shown [24,25], via the Kalman-Yakubovich Lemma [26,27,28] (also called the Positive Realness Lemma), that strictly positive real systems are absolute stable. This result is called the absolute stability theorem. In [29], strictly positive real systems are also shown to be hyperstable. This is known as the hyperstability theorem. These two theorems have proven to be invaluable tools in finite dimensional system analysis with applications ranging from nonlinear control, adaptive control to robustness analysis.

In finite dimensional analysis, positive realness is stated as a frequency domain non-negativity condition of a transfer function. By using the Kalman-Yakubovich Lemma, the frequency condition is related to a set of algebraic equations, which is called the Lur'e equations (this terminology is used in [30] for the Riccati equation; here we use it to mean the equivalent set of equations in the Kalman-Yakubovich Lemma in [31]), associated with the time domain parameters. A quadratic Lyapunov function with the interpretation of energy [32] can be constructed from the solution of the Lur'e equations. The strict decreasing property of the Lyapunov function under dissipative feedback connection then leads to the absolute and hyperstability theorems.

If the above Lyapunov method is applied here, a problem arises: The energy function associated with a positive real system is not a true Lyapunov function candidate in general. As a result, energy decaying to zero does not always imply internal (state space) stability. To circumvent this problem, our approach is to first show L_2 boundedness of the state trajectory from the Lyapunov analysis and then infer its asymptotic convergence to zero by using a generalization of the Datko's theorem [33].

In contrast to the finite-dimensional case, we define positive realness in terms of the solvability of the Lur'e equations since it is used in to the stability analysis. Sufficient conditions in terms of input/output properties and in the frequency domain (by using a Hilbert space generalization of the Positive Realness Lemma [30,34]) are also stated. The "closeness" of a system to positive realness can be characterized by a scalar index, called the ν-index for convenience, that is defined in either the time or frequency domain. (The frequency domain ν-index has been introduced in [18,19] in the context of controller design for flexible structures.) Note that both gain and phase information of the system is captured by this index. The ν-index can be computed from the finite dimensional approximation of the time domain parameters (the A, B, C, D operators). We will show that the strong convergence of the approximate parameters implies the convergence of the ν-index.

With the generalization of the Datko's Theorem and a suitable definition of positive realness, our solution of the absolute stability and hyperstability problems can be succinctly stated:

If the ν-index of a system is negative, then the system is both absolute stable and hyperstable.

By using loop transformations, the absolute stability and hyperstability results are used to analyze stability of more general systems. An acceptable class of Δ is related to the ν-index of the transformed $\mathcal{T}$. This result can be interpreted as a generalization of the circle criterion [24]. This generalization is similar to the past results on the circle criterion for evolution equations [35,36], but the internal (state space) stability result and the simple graphic test proposed here are unique to our approach. As a special case, we also recover the small gain stability criterion based on the H_∞-norm.

Absolute stability and hyperstability for evolution systems in Hilbert space has been of considerable interest in the literature [30,34,37,38,39]. Our stability analysis is different from the past approaches and our framework allows more general systems, for example, systems with multiple inputs and outputs, dynamical dissipative feedback systems, and systems that satisfy circle criterion. In particular, the absolute stability results in [34,39] are special cases of the results here (see Section 5).

When Δ contains additional structure such as diagonality, the multiplier method [§VI.9 in 40] can be used to improve the sharpness of the robustness margin. Two types of multipliers are considered, corresponding to the Popov criterion and off-circle criterion. Finding the optimal multiplier within these classes is shown to be globally convex, thus can be performed efficiently. For a more general class of multipliers, finding the optimal multiplier involves a constrained optimization problem in the unit ball in $L_2(-\infty,\infty)$. We propose an approximate finite dimensional solution by using an orthonormal basis for $L_2(-\infty,\infty)$, though the numerical aspect of this approach remains to be explored. When Δ is both diagonal and constant, a robustness margin can be computed for each quadrant of the parameter space. Specifically, if there are m diagonal elements in Δ, then we can compute 2^m robustness margins for each combination of the signs of the diagonal elements. This directional robustness information may be useful in pointing to the direction to change the operating point to enhance robustness.

For nonlinear $\mathcal{T}$'s, there are two ways to find robustness margins. One can lump the nonlinear dynamics with the perturbation Δ and then apply the results here. The bound in general will be conservative since the knowledge of the nonlinear dynamics is not explicitly used. However, we are able to recover some stability results on nonlinear systems that appeared in [41]. If the nonlinear dynamics is linear in input (i.e., linear with respect to the output of the feedback system), then one may use the nonlinear definition of passivity [42] directly. This avenue remains to be fully explored, however.

Though most of the results in this report deals with the stability analysis problem, the passivity-based stability criteria are useful in the robust control context, also. The problem of finding compensator to achieve certain desired passivity in a specified input/output channel can be transformed to an equivalent H_∞-optimization problem [43]. In the case of additive plant perturbations, an analytic bound of the achievable ν-index with respect to the additive channel can be derived. A stabilizing compensator can then be designed by solving the Nehari problem (the so-called one-block problem [44]). This approach is similar to that in [45], except a passivity-based stability criterion is used instead of the small gain criterion. Our result has the interesting feature that only the unstable portion of the open loop plant, denote it by P_u, needs to be modeled for the compensator design (resulting in a low order compensator), if the minimum Hankel singular value of P_u^* is sufficiently large. Application of this result to infinite dimensional evolution systems results in a design algorithm for stabilizing finite dimensional compensators. The numerical aspect of this algorithm has not yet been fully explored.

Following examples are provided to illustrate various aspect of the robustness analysis tools discussed in this report:

1. Robustness of the linear quadratic regulator (with full state feedback) and linear quadratic gaussian controller from the passivity viewpoint.
2. Stabilization and ν–index computation for the heat equation.
3. Two diagonal perturbation problems.
4. Three examples on the use of multipliers

The report is organized as follows. Section 2 defines positive realness in Hilbert space and states various time domain, frequency domain and input/output space conditions for positive realness. Some important lemmas needed for stability analysis in Hilbert space are given in section 3. Sections 4 and 5 present the infinite dimensional version of the absolute stability and hyperstability theorems, respectively, and their generalizations. Section 6 applies the absolute stability and hyperstability results to the robustness analysis of sector–bounded perturbations. Connection is drawn between sector–bounded perturbation and perturbation with norm upperbound and innerproduct lower bound. Application of the multiplier technique to diagonally structured feedback systems is discussed in section 7. Section 8 examines the robustness analysis for nonlinear systems. Synthesis by using the H_∞ optimization method is discussed in section 9. Stabilizing finite dimensional compensator design for evolution systems is presented as a special case. Finally, several examples are given in section 10 to illustrate application of the passivity approach to robustness analysis in this report. Proofs of the main results are included in the main text. Proofs of supportive results are given in the appendix section.

The usual notations of ≥ 0 and > 0 are used to denote positive semidefiniteness and positive definiteness of matrices, respectively. The symbols $\lambda_{\min}(A)$, $\mu_{\min}(A)$ and $\sigma_{\min}(A)$ are defined as the minimum matrix eigenvalue, minimum eigenvalue of symmetrized A (i.e., $\frac{1}{2}(A + A^T)$) and minimum matrix singular value, respectively. A coercive operator means a positive operator that is also bounded invertible in the space under consideration. The notation $\gg 0$ is used for coercivity. The space in which norms and inner products are taken will not be noted explicitly; the interpretation is inferred from the arguments. The truncated L_2 space, $L_2([0,t])$, is denoted by L_{2_t}. The inner product and norm in L_{2_t} is denoted by $\langle\cdot,\cdot\rangle_t$ and $\|\cdot\|_t$, respectively. We say $x \in L_{2_e}$, the extended L_2 space, if $x \in L_{2_t}$ for all $t \in [0,\infty)$.

The space of bounded linear operators from a Hilbert space $\mathbf{X}$ to a Hilbert space $\mathbf{Y}$ is denoted by $\mathcal{L}(\mathbf{X},\mathbf{Y})$ and $\mathcal{L}(\mathbf{X}) \triangleq \mathcal{L}(\mathbf{X},\mathbf{X})$. A C_o–semigroup $U(t)$ is said to be exponentially stable if there exists $M \geq 1$ and $\sigma > 0$ such that

$$\|U(t)\| \leq M e^{-\sigma t} \quad . \tag{1.3}$$

We say (A,B) is exponentially stabilizable and (A,C) is exponentially detectable if there exist G and K such that $A + BG$ and $A + KC$ generate exponentially C_o–semigroup , respectively. For an introduction to the C_o–semigroup approach to the study of evolution equations, see for example [1,46,47].

2. Positive Realness in Hilbert Space

The stability results in this paper are based on the positive realness of linear time invariant systems. We first define positive realness in terms of the state space parameters and then draw connections to conditions in terms of input/output signals and the frequency domain transfer function. The state space definition is useful in the later stability analysis. The frequency domain condition is convenient for computation due to the assumed finite dimensionality of the input/output spaces. The input/output condition relates positive real systems to general passive systems defined by Popov inequality.

We will introduce the ν-index to characterize "the degree of positive realness" (in a loose sense) of a given system. The ν-index is defined via the state space parameters but can be equivalently, and more conveniently, computed in the frequency domain.

2.1 Time Domain Definition of Positive Realness

We define strict positive realness, positive realness and almost positive realness for $\mathcal{T}$ in terms of the state space parameters.

Definition 1. Consider an exponentially stable system $\mathcal{T}$ given by (1.1) . If there exists $\epsilon > 0$, $P \in \mathcal{L}(\mathbf{X})$, $Q \in \mathcal{L}(\mathbf{X}, \mathbf{R}^m)$, $W \in \mathbf{R}^{m \times m}$, such that

$$(A^*P + PA + \epsilon I + Q^*Q)x = 0 \qquad \text{for all } x \in \mathcal{D}(A) \tag{2.1a}$$

$$B^*P = C - W^*Q \tag{2.1b}$$

$$W^*W = D + D^* \quad , \tag{2.1c}$$

then $\mathcal{T}$ is said to be *strictly positive real.*

If $(A, B, C, D + dI)$ is strictly positive real for all $d > 0$, then $\mathcal{T}$ is said to be *positive real.*

If (2.1) hold with $\epsilon = 0$, then $\mathcal{T}$ is said to be *almost strictly positive real.*

■

Equations (2.1 a–c) are called the Lur'e equations associated with (1.1) .

Remarks:

1. Equation (2.1a) is called the Lyapunov equation. It has been written in an algebraic form as to draw analogy to the finite-dimensional case. The solution can be equivalently and more conveniently written as

$$Px = \int_0^\infty U^*(\eta)(\epsilon I + Q^*Q)U(\eta)x \, d\eta \quad . \tag{2.2}$$

We now show that this integral is a well defined Bochner integral [46,§V.5–6]. Since $\mathcal{D}(A)$ is dense in xx [1, Corollary 2.5], given $x \in \mathbf{X}$, for all $\epsilon > 0$, there exists $z \in \mathcal{D}(A)$ such that $\|x - z\| < \epsilon$. Now, for $z \in \mathcal{D}(A)$, $U(t)z$ is continuous in t [1, Theorem 2.4]. Therefore, $\langle w, U(t)z \rangle$ is Lebesgue measurable for all $w \in \mathbf{X}$, meaning that $U(t)z$ is weakly measurable [46, Definition V.4.1]. If we assume $\mathbf{X}$ is separable, then Theorem V.4 in [46] can be used to conclude that $U(t)z$ is strongly measurable which means that it is the strong limit of a sequence of $\mathbf{X}$-valued simple functions in $[0, \infty)$. Combining the above arguments, we conclude that $U(t)x$ is also the strong limit of a sequence of simple functions. Hence, $U(t)x$ is strongly measurable [46, Definition V.4.1]. By an identical argument, we can show that the integrand in (2.2) is also strongly measurable. Since $U(t)$ is an exponentially stable C_o–semigroup , the norm of the integrand is integrable. By Theorem V.5.1 in [46], it follows that the integral in (2.2) is a well defined Bochner integral.

The exponential stability of $U(t)$ also implies that P given in (2.2) is a bounded operator. Furthermore, it is a unique solution of (2.1a) [34, Lemma 1]. Since $\epsilon I + Q^*Q$ is coercive, P is positive [34, Lemma 1]. However, P is coercive if and only if A generates a C_o -group [48]. The C_o -group assumption holds in finite dimensions but is restrictive for infinite dimensional systems, for example, the heat equation and the damped wave equation (with the damping term of the form $\frac{\partial^3}{\partial t \partial x^2}$) do not generate C_o -group. Therefore, we will not impose this requirement.

2. Unlike its finite dimensional counterpart, Definition 1 is stated in the time domain rather than the frequency domain. This is a reasonable choice since only Lur'e equations are used in the stability analysis. The frequency domain condition, which will be discussed below, can be considered as a practical way to verify the positive realness property.

3. Associated with the Lur'e equations is an energy function

$$V(x) \triangleq \langle Px, x \rangle \quad . \tag{2.3}$$

We will use this energy function extensively to deduce stability properties of the interconnected system in Fig. 1.

■

In the definition below, we introduce an index that characterizes the "degree of positive realness" for systems of the form (1.1) .

Definition 2. The ν-index of a linear time invariant system given by (2.1) is defined as

$$\nu(\mathcal{T}) = \inf\{\lambda \in \mathbf{R} : (A, B, C, D + \lambda I) \text{ is strictly positive real }\}$$

■

The relationship between ν–index and positive realness is an obvious one.

Fact 1. Given an exponentially stable system $\mathcal{T}$ as in (1.1) , $\nu(\mathcal{T}) \leq 0$ if and only if $\mathcal{T}$ is positive real.

■

2.2 Relationship between Positive Realness and Input/Output Conditions

It is well known that finite–dimensional positive real systems satisfy an input/output dissipativity condition, called the Popov inequality [49,50]. We will show in this section that a similar relationship also exists for systems described by (1.1) . First, we define the Popov inequality and the exponential Popov inequality.

Definition 3. A dynamical system with input u and output y is said to satisfy the *Popov inequality* if there exists a positive constant ξ such that for all $t \geq 0$,

$$\int_0^t y^T(s)u(s)\,ds \geq -\xi \quad . \tag{2.4}$$

The system is said to satisfy the *exponential Popov inequality* if there exist positive constants, ξ and γ such that for all $t \geq 0$,

$$\int_0^t e^{\gamma s} y^T(s)u(s)\,ds \geq -\xi \quad . \tag{2.5}$$

■

For a physical motivation of how the Popov inequality relates to passivity, consider a network with voltage as input and current as output. The total energy delivered to the network from time 0 to t is $\int_0^t u^T(s)y(s)\,ds$ [32]. If the network has zero initial energy and satisfies the Popov inequality, then energy is always delivered to the system; hence, the network either conserves or dissipates energy, or, is, in other words, passive.

To show the connection between the Popov inequality and positive realness, we shall need the continuous differentiability of the energy function (2.3) along the solution trajectory (1.2), a sufficient condition for which is stated in the following lemma.

Lemma 1. Given $x(t)$ as in (1.2), $U(t)$ exponentially stable (i.e., $U(t)$ satisfies (1.3)), P defined by

$$Px = \int_0^\infty U^*(s)RU(s)x\,ds \quad \text{for all } x \in \mathbf{X} \quad , \quad R > 0 \quad ,$$

and V as defined by

$$V = \langle Px, x\rangle \quad . \tag{2.6}$$

If $u \in L_{2_e}$, then $V(x(t))$ is differentiable in t for all $t \in [0,\infty)$ and $\dot V \triangleq \dfrac{dV(x(t))}{dt}$ is given by

$$\dot V(t, x(t)) = -\langle Rx(t), x(t)\rangle + 2\langle PBu(t), x(t)\rangle \quad . \tag{2.7}$$

Proof: The proof is given in Appendix I .

■

The relationship between positive real systems and their input/output properties can now be stated:

Proposition 1. Given $\mathcal{T}$ as in (1.1), assume that the input is in the extended L_2 space, i.e., $u \in L_{2_e}$. Then the following statements are true:

1. If $\mathcal{T}$ is almost strictly positive real then $\mathcal{T}$ satisfies the Popov inequality.
2. If $\mathcal{T}$ is strictly positive real then $\mathcal{T}$ satisfies the exponential Popov inequality.

Proof:

1. Let $V(x)$ be defined as in (2.6) .

By Lemma 1, $V(x(t))$ is differentiable along the solution and $\dot V$ is given by: (Note that V does not depend on t explicitly, but $\dot V$ may depend on t due to the external input $u(t)$.)

$$\begin{aligned}
\dot V(t, x(t)) &= -\langle Q^*Qx(t), x(t)\rangle + 2\langle PBu(t), x(t)\rangle \\
&\quad \text{(by (2.1 a) with } \epsilon = 0) \\
&= -\|Qx(t)\|^2 + 2\langle u(t), Cx(t)\rangle - 2\langle QWu(t), x(t)\rangle \\
&\quad \text{(by (2.1 b))} \\
&= -\|Qx(t)\|^2 + 2\langle u(t), y(t)\rangle - \|Wu(t)\|^2 - 2\langle Wu(t), Q^*x(t)\rangle \\
&\quad \text{(by (2.1) and (2.1 c))} \\
&= 2\langle u(t), y(t)\rangle - \|Qx(t) + Wu(t)\|^2 \\
&\leq 2\langle u(t), y(t)\rangle \quad .
\end{aligned}$$

Integrate both sides from 0 to t, then

$$\int_0^t \langle u(s), y(s)\rangle\, ds = \frac{1}{2}V(x(t)) - \frac{1}{2}V(x(0))$$
$$\geq -\frac{1}{2}V(x(0)) \quad .$$

Hence, T satisfies the Popov inequality.

2. Let $V(t,x) = e^{\gamma t} < Px, x >$. From part 1 of the proof, the derivative of V along solution is

$$\begin{aligned}\dot{V}(t,x(t)) &= \gamma V(t,x(t)) - e^{\gamma t}\|Qx(t) + Wu(t)\|^2 - \epsilon e^{\gamma t}\|x(t)\|^2 + 2e^{\gamma t}\langle u(t), y(t)\rangle \\ &\leq -(\frac{\epsilon}{\|P\|} - \gamma)V(t,x(t)) + 2e^{\gamma t}\langle u(t), y(t)\rangle \quad .\end{aligned}$$

Choose γ so that $0 < \gamma < \frac{\epsilon}{P}$. Then

$$\dot{V}(t,x(t)) \leq 2e^{\gamma t}\langle u(t), y(t)\rangle \quad .$$

By integrating both sides of the inequality, it follows that T satisfies the exponential Popov inequality.

■

2.3 Relationship between Positive Realness and Frequency Domain Conditions

Except for special cases, the Lur'e equations are difficult to verify for a given system even in finite dimensions. Infinite dimensionality only compounds the problem. On the other hand, the finite, though possibly irrational, frequency domain transfer matrix is more amenable to computation, both numerically and experimentally. In this section, we will use the Hilbert space version of the Positive Realness Lemma to derive frequency domain conditions for positive realness. The generalization of the Positive Realness Lemma to Hilbert space was first done in [30] and later restated in [34]. To state the result here, consider a quadratic form F defined on the Hilbert space $\mathbf{X} \times \mathbf{R}^m$ by

$$F(x,u) = \langle F_1 x, x\rangle + 2\text{Re}\,\langle F_2 x, u\rangle + \langle F_3 u, u\rangle \quad , x \in \mathbf{X}, \quad u \in \mathbf{R}^m \tag{2.8}$$

where $F_1 \in \mathcal{L}(\mathbf{X})$ and $F_3 \in \mathbf{R}^{m \times m}$ are self-adjoint and $F_2 \in \mathcal{L}(\mathbf{X}, \mathbf{R}^m)$. For complex vectors x and u, inner product and the linear operators in (2.8) are interpreted in the corresponding complexified spaces [34]. Specifically, given $z_{c_i} = x_i + jy_i$, $i = 1,2$, x_i, y_i, are elements in a real Hilbert space, the inner product between z_{c_1} and z_{c_2} is defined as

$$\langle z_{c_1}, z_{c_2}\rangle = (\langle x_1, x_2\rangle + \langle y_1, y_2\rangle) + j(\langle y_1, x_2\rangle - \langle x_1, y_2\rangle) \quad ,$$

and the complexification of a linear operator E as

$$Ez_{c_1} = Ex_1 + j\,Ey_1 \quad .$$

We first state the Hilbert space version of the Positive Realness Lemma in the same form as in [30,34].

Theorem 1. [30,34]

Consider an exponentially stable evolution system in a Hilbert space, $\mathcal{T}$, as described by (1.1). If for some $\delta > 0$,

$$F((j\omega I - A)^{-1}Bz, z) \geq \delta \|z\|^2, \quad \text{for all } z \in \mathbf{R}^m, \omega \in \mathbf{R} \quad , \tag{2.9}$$

then there exist $H \in \mathcal{L}(\mathbf{X})$, H self-adjoint, and $h \in \mathcal{L}(\mathbf{X}, \mathbf{R}^m)$ such that

$$2\langle Ax + Bu, Hx\rangle + F(x,u) = \|F_3^{\frac{1}{2}}(u - hx)\|^2, \quad \text{for all } (x,u) \in \mathcal{D}(A) \times \mathbf{R}^m \tag{2.10}$$

$$h = -F_3{}^{-1}(B^* H + F_2) \quad . \tag{2.11}$$

■

To put Theorem 1 in a form suitable for our use, we need to first make the following observations. By the Laplace transform identity [1],

$$(j\omega I - A)^{-1}x = \int_0^\infty e^{-j\omega t}U(t)x \, dt \quad . \tag{2.12}$$

An exponentially stable $U(t)$ implies that $U(t)x \in L_1([0,\infty); \mathbf{X})$. Hence, by the Riemann-Lebesgue Lemma, $(j\omega I - A)^{-1}x \to 0$ as $\omega \to \infty$. Therefore, (2.9) implies that F_3 is a positive definite matrix. By taking the norm of both sides of (2.12), it is straightforward to show that

$$\|(j\omega I - A)^{-1}\| \leq \frac{M}{\sigma} \quad , \tag{2.13}$$

where M and σ are related to the exponential bound of $\|U(t)\|$, as defined in (2.3).

We can now state Theorem 1 in a form that relates a frequency condition to strict positive realness.

Corollary 1. Given an exponentially stable system $\mathcal{T}$ as in (1.1), let T be the transfer function representation of $\mathcal{T}$:

$$T(j\omega) \triangleq D + C(j\omega I - A)^{-1}B \quad . \tag{2.14}$$

If there exists $\epsilon > 0$ such that

$$\text{Re}\,\langle T(j\omega)z, z\rangle \geq \epsilon \|z\|^2 \quad , \tag{2.15}$$

for all $z \in \mathbf{C}^m$ and $\omega \in \mathbf{R}$, then $\mathcal{T}$ is strictly positive real.

Proof: Equation (2.15) can be manipulated as follows:

$$\begin{aligned}
& 2\text{Re}\,\left\langle C(j\omega I - A)^{-1}Bz, z\right\rangle + 2\langle Dz, z\rangle \geq 2\epsilon \|z\|^2 \\
\Rightarrow \quad & -\eta \|(j\omega I - A)^{-1}Bz\|^2 + 2\text{Re}\,\left\langle C(j\omega I - A)^{-1}Bz, z\right\rangle + 2\langle Dz, z\rangle \geq 2\epsilon \|z\|^2 - \eta \|(j\omega I - A)^{-1}Bz\|^2 \\
& \geq (2\epsilon - \eta \frac{M^2}{\sigma^2} \|B\|^2) \|z\|^2 \\
& \text{(by (2.13))} \quad .
\end{aligned}$$

Let $0 < \eta < \frac{2\epsilon\sigma^2}{M^2\|B\|^2}$, then the above inequality implies that $F((j\omega I - A)^{-1}Bz, z) \geq \delta \|z\|^2$, $\delta > 0$, where F_1, F_2, F_3 in F are given by

$$F_1 = -\eta I \quad , \quad F_2 = C \quad , \quad F_3 = D + D^T \quad .$$

Then by Theorem 1, there exists H and h such that (2.10) and (2.11) are satisfied. As noted earlier, $F_3 > 0$, therefore, $D + D^T$ can be factorized as

$$D + D^T = W^T W \quad ,$$

which is (2.1c). Now, by setting $u = hx$ in (2.10) , we have

$$\begin{aligned} 0 &= \langle (HA + A^*H)x, x\rangle + 2u^T(B^*H + C)x - \eta\|x\|^2 + \|Wu\|^2 \\ &= \langle (HA + A^*H)x, x\rangle - \|Whx\|^2 - \eta\|x\|^2 \quad , \text{ for all } x \in \mathcal{D}(A) \\ &\quad (\text{from } (2.11)) \quad . \end{aligned} \tag{2.16}$$

Define P, Q as follows

$$P = -H \quad , \quad Q = -Wh \quad .$$

Then (2.16) implies (2.1a). Equation (2.11) can now be written as

$$B^*P - C = W^T Wh = -W^T Q \quad ,$$

which is (2.1b). Since the Lur'e equations have a solution, T is strictly positive real.

■

In section 2.1, the ν–index has been introduced as a time domain distance measure (in a heuristic and not a rigorous mathematical sense) of a system to positive realness. Based on the frequency coercivity condition (2.15) , a frequency domain measure of positivity can be defined as the uniform lower bound of T. This quantity is defined below as the ν_F–index.

Definition 4. The realness function of a complex–valued matrix $T(s) : \mathbf{C}^m \rightarrow \mathbf{C}^m$ analytic in the closed right half complex plane is defined as

$$[\mathrm{RF}(T)](\omega) = \inf_{\substack{z \in \mathbf{C}^m \\ \|z\| = 1}} \mathrm{Re}\ \langle T(j\omega)z, z\rangle \quad . \tag{2.17}$$

Negation of the infimum of the realness function is defined as the ν_F-index :

$$\nu_F(T) = -\inf_{\omega \in \mathbf{R}} [\mathrm{RF}(T)](\omega) \tag{2.18}$$

■

At each ω, $[\mathrm{RF}(T)](\omega)$ can be easily computed:

$$\begin{aligned} [\mathrm{RF}(T)](\omega) &= \mu_{\min}[T(j\omega)] \\ &= \lambda_{\min}\left[\frac{1}{2}(T(j\omega) + T^*(j\omega))\right] \quad . \end{aligned} \tag{2.19}$$

If $T(j\omega)$ is a scalar, then

$$[\mathrm{RF}(T)](\omega) = \mathrm{Re}\, T(j\omega) \quad .$$

We shall also need the definition of H_∞–norm. Let $T(s)$ be a complex valued matrix, analytic in the closed right half complex plane. Then

$$\|T\|_{H_\infty} \triangleq \sup_\omega \|T(j\omega)\|_2 \quad , \tag{2.20}$$

where $\|\cdot\|_2$ denotes the matrix 2–norm.

If $T(j\omega)$ is defined by (2.14) , it is easy to show that it is uniformly bounded and continuous in ω [40]. Hence, $\nu_F(T)$ is well defined.

Some useful properties of ν_F are summarized below.

Fact 2. Let G , H , F be $m \times m$ proper transfer matrices for exponentially stable systems of the form (1.1) . Then the following statements are true.

1. $\nu_F(cI) = -c \quad , \quad c = \text{constant}$.
2. $\nu_F(\alpha G) = \begin{cases} \alpha \nu_F(G) & \text{if } \alpha > 0 \\ -\alpha \nu_F(-G) & \text{if } \alpha < 0 \end{cases}$.
3. $\nu_F(G+H) \leq \nu_F(G) + \nu_F(H)$.
4. $\nu_F(cI+G) = \nu_F(G) - c$.
5. $\nu_F(G) \leq \|G\|_{H_\infty}$.

   $\nu_F(-G) \leq \|G\|_{H_\infty}$.
6. G strictly proper implies $\nu_F(G) \geq 0$.

   (Strict properness of G means $\lim_{|s| \to \infty} G(s) \to 0$.)
7. If the internal parameters of G are (A, B, C, D), then $\nu_F(G) < 0 \Rightarrow D > 0$.
8. $\nu_F(K^* G K) \leq \nu_F(G) \sigma^2_{\min}(K)$ for any complex matrix K .
9. If $\frac{1}{2}(G + G^*) = H^* F H$, then $\nu_F(G) \leq \nu_F(F) \inf_{\omega} \sigma^2_{\min}(H(j\omega))$.
10. $\sup_{K \text{ unitary}} \nu_F(GK) = \sup_{K \text{ unitary}} \nu_F(KG) = \|G\|_{H_\infty} = \sup_{K_1, K_2 \text{ unitary}} \nu_F(K_1^* G K_2)$.
11. If G is block diagonal with square diagonal blocks $\{G_i\}$, then

    $\nu_F(G) = \max_i \nu_F(G_i)$.
12. ν_F is Lipschitz continuous in the H_∞ norm topology with Lipschitz constant $= 1$.

Proof: The proof is given in Appendix II .

■

The most useful aspect of the ν_F–index is its connection to positive realness and the ν–index. This result is summarized below.

Proposition 2. Given an exponentially stable system T as described by (1.1) . Let $T(s)$ be its transfer function. The following statements are true.

1. If $\nu_F(T) < 0$ then T is strictly positive real.

2. If T is strictly positive real and $D > 0$ then $\nu_F(T) < 0$.

3. T is positive real if and only if $\nu_F(T) \leq 0$.

4. If T is almost strictly positive real then $\nu_F(T) \leq 0$.

5. $\nu_F(T) = \nu(T)$.

Proof:

1. This fact follows from Definition 4 and Corollary 1.

2. Assume T is strictly positive real. Compute the Hermitian part of the transfer function as follows:

$$
\begin{aligned}
& T(j\omega) + T^*(j\omega) \\
=& D + D^T + C(j\omega I - A)^{-1}B + B^*(-j\omega I - A^*)^{-1}C^* \\
=& W^T W + (B^* P - W^T Q)(j\omega I - A)^{-1}B + B^*(-j\omega I - A^*)^{-1}(PB - Q^* W) \\
& \quad (\text{ by } (2.1c)\ ) \\
=& W^T W + B^*(-j\omega I - A^*)^{-1}\left[(-j\omega I - A^*)P + P(j\omega I - A)\right](j\omega I - A)^{-1}B \\
& \quad - W^T Q(j\omega I - A)^{-1}B - B^*(-j\omega I - A^*)^{-1}Q^* W \\
=& W^T W + B^*(-j\omega I - A^*)^{-1}(Q^*Q + \epsilon I)(j\omega I - A)^{-1}B - W^T Q(j\omega I - A)^{-1}B - B^*(-j\omega I - A^*)^{-1}Q^* W \\
& \quad (\text{ by } (2.1a)\ ) \\
=& (W^T - B^*(-j\omega I - A^*)^{-1}Q^*)(W - Q(j\omega I - A)^{-1}B) + \\
& \quad \epsilon B^*(-j\omega I - A^*)^{-1}(j\omega I - A)^{-1}B \geq 0 \quad .
\end{aligned} \tag{2.21}
$$

This implies $\nu_F(T) \leq 0$. Assume $\nu_F(T) = 0$. Then there exist $\{u_n\} \subset \mathbf{C}^m$, $\|u_n\| = 1$, and $\{\omega_n\}$ such that

$$0 \leq \langle (T(j\omega_n) + T^*(j\omega_n))\, u_n, u_n \rangle \leq \frac{1}{n} \quad . \tag{2.22}$$

We first show that $\{\omega_n\}$ is a bounded sequence. Assume the contrary, i.e., assume some subsequence $\omega_n \to \infty$ as $n \to \infty$. Then the inner-product in (2.22) converges to $\langle Du_n, u_n \rangle$ and its upperbound converges to zero. Since $D > 0$ by assumption, this is a contradiction. Hence, $\{u_n\}$ and $\{\omega_n\}$ are both bounded sequences and therefore contain convergent subsequences $\{u_{n_k}\}$ and $\{\omega_{n_k}\}$. Let their limits be u_o and ω_o. Then

$$\langle (T(j\omega_o) + T^*(j\omega_o))\, u_o, u_o \rangle = 0 \quad .$$

It follows from (2.21) that

$$
\begin{aligned}
W u_o - Q(j\omega_o I - A)^{-1} B u_o &= 0 \\
(j\omega_o I - A)^{-1} B u_o &= 0 \quad .
\end{aligned}
$$

Substituting the second equality into the first yields

$$W u_o = 0 \quad . \tag{2.23}$$

By assumption, $D > 0$, which implies $W > 0$; hence, (2.23) implies $u_o = 0$. This is a contradiction, since $\|u_{n_k}\| = 1$ and $u_{n_k} \to u_o$. It follows then $\nu_F(T) < 0$.

3. Assume $\nu_F(T) \leq 0$. The transfer function of $(A, B, C, D + \lambda I)$ is $T(j\omega) + \lambda I$. By part 4 in Fact 2, $\nu_F(T + \lambda I) < 0$ for all $\lambda > 0$. Hence, T is positive real by Definition 2 and part 1 of this proposition. Now assume T is positive real. By Definition 2 and part 2 of this proposition, there exists a monotonically decreasing sequence $\{\eta_n\}$, $\eta_n > 0$, such that for all $z \in \mathbf{C}^m$,

$$z^*(T(j\omega) + T^*(j\omega) + \frac{1}{n})z \geq \eta_n \|z\|^2 \quad .$$

As $n \to \infty$, $\eta_n \to \eta \geq 0$ and $\frac{1}{n} \to 0$. Hence,

$$z^*(T(j\omega) + T^*(j\omega))z \geq 0 \quad ,$$

for all $z \in \mathbf{C}^m$. This implies that $\nu_F(T) \leq 0$.

4. Assume (2.1) holds with $\epsilon = 0$. Then $\nu_F(T) \leq 0$ follows from (2.21) in the proof of part 2.

5. First note that an exponentially stable system remains exponentially stable with any constant feedforward. From part 4 in Fact 2, $\nu_F(\mathcal{T} + \nu_F(T) \cdot I) = 0$. This implies $\mathcal{T} + \nu_F(T) \cdot I$ is positive real by part 3. By Fact 1, $\nu(\mathcal{T} + \nu_F(T) \cdot I) \leq 0$. It is easy to show that $\nu(\mathcal{T} + \nu_F(T) \cdot I) = \nu(T) - \nu_F(T)$. Hence, $\nu(T) \leq \nu_F(T)$.

The reverse inequality follows from:

$$
\begin{aligned}
&\nu(T) \cdot I + \mathcal{T} \ \text{ is positive real (by Definition 2).} \\
\Rightarrow &\nu_F(\nu(T) \cdot I + T) \leq 0 \ \text{ by part 3.} \\
\Rightarrow &\nu_F(T) - \nu(T) \leq 0 \ \text{ by part 4 in Fact 2.} \\
\Rightarrow &\nu_F(T) \leq \nu(T) \quad .
\end{aligned}
$$

Combining the results above, we have $\nu(T) = \nu_F(T)$.

■

In the transfer matrix representation of system (1.1) , the $\{i,j\}$ element of the transfer matrix is $\langle c_i, (j\omega I - A)^{-1} b_k \rangle$. The computation of the ν_F–index then involves solving an integro–differential equation of the form

$$
(j\omega I - A)z = b \quad , z \in \mathcal{D}(A) \quad ,
$$

for some given finite set of b's $\in \mathbf{X}$ and for each ω. An approximate numerical solution can be used to obtain an estimate of the ν_F–index. Another approach is to approximate $\mathcal{T}$ with some finite dimensional system and compute the ν_F–index of the approximate system. The following result relates the convergence of a sequence of such approximate systems to the convergence of their ν_F–indices.

Proposition 3. Suppose the internal parameters (A, B, C, D) are approximated by (A_n, B_n, C_n, D_n) where A_n is exponentially stable for each n and $A_n \rightarrow A$, $B_n \rightarrow B$, $C_n \rightarrow C$ and $D_n \rightarrow D$ strongly (since the input/output spaces are finite dimensional, the convergence of B, C and D are actually in norm). Let T_n be the transfer function associated with (A_n, B_n, C_n, D_n). Then $\mathrm{RF}(T_n)(\omega) \rightarrow \mathrm{RF}(T)(\omega)$ uniformly for all $\omega \in \Omega$ where Ω is any compact set in $\mathbf{R}$, as $n \rightarrow \infty$.

Proof: The proof is given in Appendix III .

■

By selecting a frequency range $\Omega = [-N, N]$ for N large enough, $\inf_\Omega \mathrm{RF}(T)(\omega)$ can be made arbitrarily close to $\nu_F(T)$. Then by Proposition 3, $\inf_\Omega \mathrm{RF}(T_n)(\omega)$ will converge arbitrarily close to $\nu_F(T)$, also. Hence, in practice, an approximation of $\nu_F(T)$ can be obtained by constructing a sequence of approximate systems $\{T_n\}$ (perhaps finite dimensional) whose ν_F–indices can be more easily computed. An example comparing these two approaches of computing $\nu_F(T)$ is discussed in Section 8.

2.4 Finite Dimensional Positive Real Systems

Stronger connections between various state space, input/output and transfer function conditions on positive realness can be shown for finite dimensional systems. First we restate a theorem from [51] which provides a list of related conditions on strict positive realness. These conditions can be organized into three tiers of necessary and sufficient conditions of increasing strength. In the top tier are the frequency coercivity condition and two input/output coercivity conditions. They imply strict positive realness, and if $D > 0$, they are in fact equivalent. The second tier, which is implied by the top tier, consist of two state space conditions involving particular forms of the Lur'e equations, two frequency domain positivity conditions and two input/output conditions one of which is the exponential Popov inequality. They all are sufficient for strict positive realness and are necessary when $D = 0$. At the bottom tier lies a lone frequency positivity condition which has been erroneously stated in [50,32] as a sufficient condition for the solvability of the Lur'e equations. It is weaker than the previous two, but does not imply strict positive realness in general.

Theorem 2.

Let T denote an exponentially stable linear time invariant system with state space parameters (A, B, C, D) and transfer function $T(s)$. Assume $\sigma_{\min}(B) > 0$. Consider the following statements:

1. T is strictly positive real.

1'. Same as 1. except L is related to P by

$$L = 2\mu P \tag{2.24}$$

for some $\mu > 0$.

2. There exists $\eta > 0$ such that for all $\omega \in \mathbf{R}$

$$T(j\omega) + T^*(j\omega) \geq \eta I \tag{2.25}$$

3. For all $\omega \in \mathbf{R}$

$$T(j\omega) + T^*(j\omega) > 0 \tag{2.26}$$

4. For all $\omega \in \mathbf{R}$

$$T(j\omega) + T^*(j\omega) > 0 \tag{2.27a}$$

and

$$\lim_{\omega \to \infty} \omega^2 (T(j\omega) + T^*(j\omega)) > 0 \tag{2.27b}$$

5. The system T can be realized as the driving point impedance of a multiport dissipative network.

6. The Lur'e equations with $L = 0$ are satisfied by the internal parameter set $(A + \mu I, B, C, D)$ corresponding to $T(j\omega - \mu)$ for some $\mu > 0$.

7. For all $\omega \in \mathbf{R}$, there exists $\mu > 0$ such that

$$T(j\omega - \mu) + T^*(j\omega - \mu) \geq 0 \tag{2.28}$$

8. There exist positive constants ρ and ξ such that for all $T \geq 0$

$$\int_0^T u^T(t) y(t)\, dt \geq \xi + \rho \int_0^T \|u(t)\|^2\, dt \tag{2.29}$$

9. There exist positive constants γ and ξ, such that for all $T \geq 0$

$$\int_0^T e^{\gamma t} u^T(t) y(t)\, dt \geq \xi \tag{2.30}$$

10. There exists a positive constant α such that the following kernel is positive in $L_2(\mathbf{R}_+; \mathbf{R}^{m \times m})$

$$K(t-s) = D\delta(t-s) + Ce^{(A+\alpha I)(t-s)} B.1(t-s) \tag{2.31}$$

where δ and 1 denote the Dirac delta function and the step function, respectively.

11. The following kernel is coercive in $L_2([0,T]; \mathbf{R}^{m \times m})$, for all T.

$$K(t-s) = D\delta(t-s) + Ce^{A(t-s)} B.1(t-s) \tag{2.32}$$

These statements are related as follows:

$$
(1) \quad \left\{ \begin{array}{llllll}
\Longleftarrow & (2) & \Longleftrightarrow & (8) & \Longleftrightarrow & (11) \\
\Longrightarrow & & & & & \\
(\text{if } D > 0) & & & & & \\
 & & & \Downarrow & & \\
\Longleftarrow & (1') & \Longleftrightarrow & (4) \Longleftrightarrow (5) \Longleftrightarrow (6) \Longleftrightarrow (7) \Longleftrightarrow (9) \Longleftrightarrow (10) & & \\
\Longrightarrow & & & & & \\
(\text{if } D = 0) & & & & & \\
 & & & \Downarrow & & \\
 & & & (3) & &
\end{array} \right.
$$

Proof: The proof is given in Appendix IV .

■

For SISO systems, condition (4) has been noted to be necessary for condition (5) [25] and later shown to be necessary and sufficient in [52]. Condition (8) was termed u–strictly–passive for nonlinear systems in [42].

For finite dimensional systems, Definition 1 is non–standard. At present, there appears to be no consensus in the literature on the definition of strict positive realness. Condition (3) has been used as a definition for strict positive realness [50,32]. As seen in Theorem 2, it is in general too weak to be used for stability analysis. In [52], condition (5) was used as the definition for strict positive realness. If a frequency domain definition of strict positive realness is sought, condition (4) is a reasonable choice. Condition (7) was used by [25] as a definition for strict positive realness for both SISO and MIMO systems. This choice is not as appealing as condition (4), as it depends on an unknown constant μ (see (2.28)).

If the frequency condition (2.25) is satisfied, then Theorem 2 states that the Lur'e equations (2.1) associated with $\mathcal{T}$ has a solution. How do the constants, η in (2.25) and ϵ in (2.1) , relate to each other? This question is important because ϵ determines the rate of convergence in exponential stability results in the later sections but it is more difficult to obtain than η. Following corollary provides an answer.

Corollary 2. Suppose condition 2 in Theorem 2 holds. Define

$$T_1(j\omega) \triangleq C(j\omega I - A)^{-1}B \tag{2.33}$$

Then condition 1 holds with

$$L = \epsilon I + \gamma C^T C \tag{2.34}$$

where ϵ and γ are any positive constants that satisfy

$$\gamma < \frac{\eta}{\|T_1\|^2_{H_\infty}} \tag{2.35}$$

and

$$\epsilon < \frac{\eta - \gamma \|T_1\|^2_{H_\infty}}{\|(j\omega I - A)^{-1}B\|^2_{H_\infty}} \quad . \tag{2.36}$$

Proof: The proof is given in Appendix V .

∎

A list of equivalent conditions can be stated for finite dimensional positive real systems as in Theorem 2 (a similar list is also given in Theorem B.2.1 in [50]). We will sacrifice some generality by requiring $\mathcal{T}$ to be exponentially stable. The full generality can be obtained by incorporating the lossless real lemma [53].

Proposition 4. Given an exponentially stable LTI system $\mathcal{T}$ with an internal parameter set (A, B, C, D) and transfer function $T(s)$. The following statements are equivalent:

1. $\mathcal{T}$ is almost strictly positive real and P that solves the associated Lur'e equations is positive definite.
2. For all $\omega \in \mathbf{R}$
$$T(j\omega) + T^*(j\omega) \geq 0 \quad . \tag{2.37}$$
3. $\mathcal{T}$ can be realized as the driving point impedance of a multiport passive network.
4. There exists a constant ξ, such that for all $T \geq 0$,
$$\int_0^T u^T(t) y(t)\, dt \geq \xi \quad . \tag{2.38}$$
5. The following kernel is non-negative in $L_2(\mathbf{R}_+; \mathbf{R}^{m \times m})$
$$K(t-s) = D\delta(t-s) + Ce^{A(t-s)}B \,.\, 1(t-s) \quad .$$
6. $\mathcal{T}$ is positive real.

Proof: The proof is given in Appendix VI .

∎

3. Useful Lemmas

Two technical lemmas needed for later stability analysis are stated in this section. The first lemma generalizes Datko's theorem [33] to systems described by (1.1) with the input $u \in L_2([0,\infty); \mathbf{R}^m)$. The second lemma relates the negativity of the derivative of the Lyapunov function to internal stability. These two results form a powerful combination that enables one to show stability without using a coercive operator in the Lyapunov function.

The L_2-boundedness of the state trajectory, $x(t)$, is equivalent to the internal exponential stability in the zero input case [33]. The next lemma generalizes this result by showing $x(t) \to 0$ as $t \to \infty$ when the input is in L_2.

Lemma 2. Given $x(t)$ as in (1.2) :

$$x(t) = U(t)x_o + \int_0^t U(t-s)Bu(s)\,ds \quad , x_o \in \mathbf{X} \quad , \tag{1.2}$$

Assume $u \in L_2([0,\infty); \mathbf{R}^m)$. If for every $x_o \in \mathbf{X}$, there exists $K(x_o) \in \mathbf{R}$ such that

$$\int_0^\infty \|x(s)\|^2 ds \le K(x_o) < \infty \quad ,$$

then

1. $U(t)$ is an exponentially stable C_o-semigroup , and
2. $\|x(t)\| \to 0$ as $t \to \infty$ for all $x_o \in \mathbf{X}$.

Proof: The proof is given in Appendix VII .

■

As stated before, unlike the finite dimensional case, P that solves the Lyapunov equation (2.1a) is not bounded invertible in general. This means that the norm induced by the inner product $\langle x, y\rangle_1 \triangleq \langle Px, y\rangle$ is weaker than the underlying norm of the state space. Hence, convergence in $\|\cdot\|_1$ does not imply convergence in the natural norm. This prevents a direct application of the standard proofs for the absolute stability and hyperstability in the finite dimensions where the quadratic form, $\langle Px, x\rangle = \|x\|_1^2$ is used as a Lyapunov function candidate. This problem is avoided by introducing a lemma below which relates Lyapunov type of analysis to internal stability without requiring the bounded invertibility of P.

Lemma 3. Given $x(t)$ as in (1.2) . Define $V(x) \triangleq \langle Px, x\rangle$ for some $P > 0$ and bounded. Assume for all $x_o \in \mathbf{X}$, $V(x(t))$ is differentiable in t and there exists $\epsilon > 0$ such that

$$\dot{V}(t, x(t)) \le -\epsilon\,\|x(t)\|^2 \quad . \tag{3.1}$$

Then for all $\sigma \in [0, \frac{1}{2}\epsilon\,\|P\|^{-1})$ and $x_o \in \mathbf{X}$, the following statements are true:

1. If $u(t) \in L_2([0,\infty); \mathbf{R}^m)$, then $x(t) \to 0$ as $t \to \infty$.
2. $e^{\sigma t}x(t) \in L_2([0,\infty); \mathbf{X})$.
3. If $e^{\sigma t}u(t) \in L_2([0,\infty); \mathbf{R})$, then $x(t) \to 0$ exponentially with decay rate σ as $t \to \infty$.

Proof: The proof is given in Appendix VIII .

∎

4. Absolute Stability

In this section, we will show the following generalization of the finite–dimensional absolute stability theorem: $\nu(\mathcal{T}) < 0 \Rightarrow \mathcal{T}$ is absolutely stable. This result can be considered as an extension of the passivity theorem [40], which says two interconnected passive systems are input/output stable, to internal state space stability. By applying the absolute stability theorem to $\mathcal{T}$ and Δ after simple loop transformations (feedforward and feedback of both $\mathcal{T}$ and Δ by constant systems), we show that an interconnection of sector bounded $\mathcal{T}$ and Δ is stable. The interpretation of this result as a Hilbert space version of the circle criterion [24,35,36,40] will be given in Section 6.

To ensure wellposedness of the interconnected system, the following technical assumption is made throughout this section.

Assumption 1. Given the interconnected system in Fig. 1. Assume that there exists a unique solution u of

$$u = -\Delta(t, Cx + Du) \quad , \tag{4.1}$$

for every $t \geq 0$ and $x \in \mathbf{X}$. Define Δ_1 as the solution operator

$$u = -\Delta_1(t, Cx) \quad . \tag{4.2}$$

Assume Δ_1 is bounded uniformly in t, for t in bounded intervals, continuous in t and locally Lipschitz continuous with respect to Cx.

∎

The uniqueness of solution assumption in Assumption 1 is needed to remove the possibility of a multi–valued map from Cx to u, for such generality is not addressed in this paper. The boundedness, continuity and local Lipschitz conditions on Δ_1 implies that a unique mild solution of the interconnected system exists for all t such that $x(t)$ is bounded [1, Theorem 6.1.4.]. We will discuss below a condition on Δ that will assure these required properties on Δ_1.

If Δ is bounded uniformly in t, for t in bounded intervals, and continuous in t, it is easy to see that the same properties hold for Δ_1. Suppose Δ is locally Lipschitz in the second variable, i.e., for every $T \geq 0$ and constant $c \geq 0$, there is a constant $L(c, T)$ such that

$$\|\Delta(t, z_1) - \Delta(t, z_2)\| \leq L(c, T)\, \|z_1 - z_2\|$$

holds for all z_1, $z_2 \in \mathbf{R}^m$ with $\|z_1\| \leq c$, $\|z_2\| \leq c$ and $t \in [0, T]$. Let $u_i = \Delta_1(t, Cx_i)$, $i = 1, 2$, then for every $T \geq 0$ and $c \geq 0$, we have

$$\begin{aligned} &\|u_1 - u_2\| \\ =&\|\Delta_1(t, Cx_1) - \Delta_1(t, Cx_2)\| \\ =&\|\Delta(t, Cx_1 - Du_1) - \Delta(t, Cx_2 - Du_2)\| \\ \leq& L(c, T)\,(\,\|Cx_1 - Cx_2\| + \|D\|\,\|u_1 - u_2\|) \end{aligned} \tag{4.3}$$

for $t \in [0,T]$, $\|Cx_i - Du_i\| \le c$, $i = 1,2$. Given $\|Cx_i\| \le c_1$, $c_1 \ge 0$, there exists c such that $\|Cx_i - Du_i\| \le c$ if

$$\|\Delta\|\,\|D\| < 1 \tag{4.4}$$

where

$$\|\Delta\| \triangleq \sup_{t\in\mathbf{R}} \sup_{z\in\mathbf{R}^m} \frac{\|\Delta(t,z)\|}{\|z\|} \quad .$$

Condition (4.4) implies $\Delta(t,0) = 0$ and also guarantees a unique solution of (4.1) , but it is not necessary. Now, if

$$L(c,T)\,\|D\| < 1 \tag{4.5}$$

for all $T \ge 0$ and $c \ge 0$, then (4.4) is satisfied and the local Lipschitzness of Δ_1 follows from (4.3) . In many situations, $\mathcal{T}$ is strictly proper, i.e., $D = 0$, then the condition for the wellposedness of solution can be placed on Δ directly.

The following lemma states that the non-negativity of Δ implies x and $u \in L_{2_e}$, which is needed for the differentiability of V along the solution trajectory.

Lemma 4. Consider the interconnected system in Fig. 1. Assume $D > 0$ and $y^T\Delta(t,y) \ge 0$ for all $t \ge 0$ and $y \in \mathbf{R}^m$. Let u be the unique solution of (4.1) for each $t \ge 0$. Then $x(t)$ does not finitely escape (bounded on bounded intervals), $x \in L_{2_e}(\mathbf{X})$, $u \in L_{2_e}(\mathbf{R}^m)$ and there exists $\eta > 0$ such that $\|u\| \le \eta\,\|x\|$.

Proof: The proof is given in Appendix IX .

■

If Δ is non-negative, then Assumption 1 and Lemma 4 together imply that a unique mild solution exists for all $t \ge 0$.

We now state and prove the generalization of the absolute stability theorem.

Theorem 3. Given the interconnected system as in Fig. 1. Let $\mathcal{T}$ be an exponentially stable system given by (1.1) . If $\nu(\mathcal{T}) < 0$ and

$$y^T\Delta(t,y) \ge 0 \quad , \tag{4.6}$$

for all $t \ge 0$ and $y \in \mathbf{R}^m$, then $x(t) \to 0$ exponentially as $t \to \infty$.

Proof: Let $V(x) = \langle Px, x\rangle$, where P is given by the solution of the Lur'e equation (2.1) associated with $\mathcal{T}$. Now,

$$\begin{aligned} |z^T\Delta_1(t,z)| \ge &- z^T\Delta_1(t,z) \\ =& z^T\Delta(t, z + Du) \\ =& (z + Du)^T\Delta(t, z + Du) + u^T Du \\ \ge& \mu_{\min}(D)\,\|\Delta_1(t,z)\|^2 \quad \text{(by (4.6))} \quad . \end{aligned}$$

By property 7 of Fact 2, $\mu_{\min}(D) > 0$. Hence, $\Delta_1(t,0) = 0$. From Lemma 4, $u \in L_{2_e}$, which, by Lemma 1, implies that $V(x(t))$ is differentiable along the mild solution (2.2) and $\dot{V}(x(t))$ is given by

$$
\begin{aligned}
\dot{V}(t,x(t)) &= -\epsilon\,\|x(t)\|^2 - \|Qx(t)\|^2 + 2\,\langle PBu(t), x(t)\rangle \\
&= -\epsilon\,\|x(t)\|^2 - \|Qx(t)\|^2 - 2\,\langle PB\Delta(t,y(t)), x(t)\rangle \\
&= -\epsilon\,\|x(t)\|^2 - \|Qx(t)\|^2 - 2y(t)^T\Delta(t,y(t)) \\
&\qquad - 2\Delta(t,y(t))^T D\Delta(t,y(t)) + 2\Delta(t,y(t))^T W^T Qx(t) \\
&\qquad \text{(by (2.1b))} \\
&= -\epsilon\,\|x(t)\|^2 - 2y(t)^T\Delta(t,y(t)) - \|Qx(t) - W\Delta(t,y(t))\|^2 \\
&\qquad \text{(by (2.1c) and completing the square)} \\
&\le -\epsilon\,\|x(t)\|^2 \quad \text{(by (4.6))} \quad .
\end{aligned}
$$

By Lemma 3, part 2, the last inequality implies that $e^{\sigma t}x(t) \in L_2([0,\infty);\mathbf{X})$, for all $\sigma \in (0, \frac{1}{2}\epsilon\,\|P\|^{-1})$. From Lemma 4, this implies $e^{\sigma t}u(t) \in L_2([0,\infty);\mathbf{R}^m)$. By Lemma 3, part 3, $x(t)$ converges to zero exponentially with rate σ as $t \to \infty$.

∎

In many applications, the forward system $\mathcal{T}$ is strictly proper; since this implies $\nu(\mathcal{T}) \ge 0$, Theorem 3, as it stands, is not applicable. In the remainder of this section, we will generalize Theorem 3 to more general classes of systems.

The interconnected system in Fig. 1 can be transformed to an equivalent system by using a loop transformation (§III.6 in [40], §5.5 in [54]) as shown in Fig. 2. The corollary below applies Theorem 3 to the transformed interconnected system.

Corollary 3. Consider the interconnected system in Fig. 1. Assume $\mathcal{T}$ is an exponentially stable system. If

$$\nu(\mathcal{T}) < \frac{1}{\beta} \quad ,\beta > 0, \tag{4.7}$$

and Δ satisfies

$$\langle \Delta(t,y), (\Delta(t,y) - \beta y)\rangle \le 0 \quad \text{for all } t \ge 0 \text{ and } y \in \mathbf{R}^m \quad , \tag{4.8}$$

then $x(t) \to 0$ exponentially as $t \to \infty$.

Proof: The transformed forward system, $\mathcal{T} + \frac{1}{\beta}\cdot I$ is exponentially stable and $\nu(\mathcal{T} + \frac{1}{\beta}\cdot I) < 0$ by (4.7). Let $\tilde{y}$ be the input into the transformed feedback system. Then

$$\tilde{y} = y - \frac{1}{\beta}\Delta(t,y) \quad .$$

The inner product between the input and output of the transformed feedback system is

$$\tilde{y}^T\Delta(t,y) = y^T\Delta(t,y) - \frac{1}{\beta}\,\|\Delta(t,y)\|^2 \quad ,$$

which is non-negative by (4.8). By Lemma 4, the internal signals do not finitely escape. Hence, by Assumption 1, a unique mild solution exists for all $t \ge 0$. The exponential convergence of $x(t)$ to zero then follows by applying Theorem 3 to the transformed system.

∎

Fig. 1 can also be transformed to an equivalent form as in Fig. 3, in which the original forward and feedback systems are both wrapped around with positive feedback, $\alpha \cdot I$. Let v be the input of the transformed forward system. Then $u = \alpha y + v$. If $\frac{1}{\alpha} \notin \sigma(D)$ ($\sigma(D)$ denotes the spectrum of D), then a realization of the transformed forward system is

$$\dot{x} = (A + \alpha(I - \alpha D)^{-1}C)x + B(I - \alpha D)^{-1}v$$
$$y = (I - \alpha D)^{-1}(Cx + Dv) \quad .$$

Since $A + \alpha(I - \alpha D)^{-1}C$ is a bounded perturbation of the original generator A, it also generates a C_o-semigroup [Theorem 3.1.1 of 1]. Hence the transformed forward system belongs to the class of systems described by (1.1) .

The following corollary follows directly from Corollary 3.

Corollary 4. Consider the interconnected system in Fig. 1. Assume $\mathcal{T}$ is exponentially stable. Define

$$\tilde{\mathcal{T}} = (I - \alpha\mathcal{T})^{-1}\mathcal{T} \quad , \tag{4.9}$$

where $\frac{1}{\alpha} \notin \sigma(D)$. Let the transfer function representation of $\tilde{\mathcal{T}}$ be $\tilde{T}(s)$. If $\tilde{\mathcal{T}}$ is exponentially stable and

$$\nu(\tilde{T}) < \frac{1}{\beta} \quad , \beta > 0, \tag{4.10}$$

and Δ satisfies

$$\langle \Delta(t,y) + \alpha y, (\Delta(t,y) - (\beta - \alpha)y) \rangle \leq 0 \quad \text{for all } t \geq 0 \text{ and } y \in \mathbb{R}^m \quad , \tag{4.11}$$

then $x(t) \to 0$ as $t \to \infty$.

Proof: In Fig. 3, the transformed forward system is $\tilde{\mathcal{T}}$ and the transformed feedback system is $\Delta(t,y) + \alpha y$. The stated result follows from Corollary 3 by replacing $\Delta(t,y)$ by $\Delta(t,y) + \alpha y$.

■

Remarks:

4. A sufficient condition for the exponential stability requirement of $\tilde{\mathcal{T}}$ in Corollary 3 can be obtained by applying Corollary 2. If

$$\begin{aligned} &\alpha \geq 0 \quad \text{and} \quad \alpha < \frac{1}{\nu(-T)} \quad \text{or} \\ &\alpha \leq 0 \quad \text{and} \quad \alpha < \frac{1}{\nu(T)} \quad , \end{aligned} \tag{4.12}$$

then $\tilde{\mathcal{T}}$ is exponentially stable. Alternatively, the graphic Nyquist test [55] can also be used, which has the advantage of being both necessary and sufficient.

5. When $m = 1$ (single-input/single-output case), the class of Δ that satisfies (4.11) has a natural interpretation of sector-boundedness: for each t, the graph of Δ lies between two lines: $-\alpha y$ and $(\beta - \alpha)y$. For $m > 1$, we call (4.11) a general sector-boundedness condition, though the interpretation is less clear. There are two special cases worth noting, however. In the first case, if $\beta = 2\alpha$, then (4.11) reduces to a single norm upperbound of Δ by α. In the second case, if for each t, Δ is linear and symmetric ($\Delta(t,y) = \tilde{\Delta}(t)y$ and $\tilde{\Delta}$ is symmetric), then (4.11) can be replaced by a norm upperbound on $\tilde{\Delta}$ by $\beta - \alpha$ and a lowerbound on $\mu_{\min}\left(\tilde{\Delta}\right)$ by $-\alpha$ (see section 6).

6. In Theorem 3 and Corollaries 2 and 3, if Δ does not depend on t, the resulting stability of the interconnected system is uniform with respect to the initial time.

■

5. Hyperstability

We will prove the hyperstability theorem in the following form: If $\nu(\mathcal{T}) < 0$, then $\mathcal{T}$ is hyperstable and exponentially hyperstable. This result is similar to the absolute stability theorem, except the feedback system Δ is dissipative in the more general sense of the Popov inequality or exponential Popov inequality. Indeed, the absolute stability theorem can be considered as a special case. By applying the hyperstability theorem to the interconnected systems after applying the same loop transformation as in Section 4, a stability condition for general sector bounded $\mathcal{T}$ and Δ is obtained.

We restrict our analysis to the class of feedback systems that preserve the wellposedness of the overall interconnected system. This assumption is explicitly stated below.

Assumption 2. Given the interconnected system in Fig. 1. Assume a unique mild solution of $\mathcal{T}$ exists for all $t \geq 0$.

■

A class of feedback systems for which Assumption 2 is satisfied consists of evolution equations with a memoryless nonlinear feedback. Consider the following class of feedback systems

$$\dot{z} = Fz + Gy \tag{5.1a}$$

$$w = Hz + Jy \tag{5.1b}$$

$$-u = \phi(t, w) \tag{5.2}$$

where $z \in \mathbf{Z}$, $\mathbf{Z}$ is a real Hilbert space, $w \in \mathbf{R}^{\ell}$, F is the infinitesimal generator of a C_o–semigroup , G, H are bounded operators, $\phi : \mathbf{R}^{\ell} \to \mathbf{R}^{m}$ is a time–varying nonlinear function. Eq. (5.2) can be written as

$$-u = \phi(t, Hz + JCx + JDu) \quad . \tag{5.3}$$

The discussion on the wellposedness of solutions in Section 4 also applies here. Assume u in (5.3) can be uniquely solved for all t, x and z, i.e., there exists a function ϕ_1 such that

$$u = \phi_1(t, Hz + JCx) \quad . \tag{5.4}$$

Further assume that ϕ_1 is locally Lipschitz with respect to the second argument, uniformly bounded in t for t in bounded intervals, and continuous in t. As shown in Section 4, a sufficient condition for the existence of such function ϕ_1 is that ϕ is uniformly bounded in t for t in bounded intervals, and continuous in t, and ϕ is locally Lipschitz with the Lipschitz constant $L(c, T)$ that satisfies

$$L(c,T)\,\|JD\| < 1 \quad , \tag{5.5}$$

for all $T \geq 0$ and $c \geq 0$. The interconnected system can now be written in the following form:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & 0 \\ GC & F \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} - \begin{bmatrix} B \\ GD \end{bmatrix} \phi_1 \left(t, [JC \; : \; H] \begin{bmatrix} x \\ z \end{bmatrix} \right) \quad . \tag{5.6}$$

Since $A_o = \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix}$ generates a C_o–semigroup and $A_1 = \begin{bmatrix} A & 0 \\ GC & F \end{bmatrix}$ is a bounded perturbation of A_o, A_1 also generates a C_o–semigroup (Theorem 3.1.1 in [1]). The nonlinear feedback map $\phi(t, [JC \; : \; H] \cdot)$ is locally Lipschitz, bounded uniformly in t for t in bounded intervals and continuous in t, since ϕ has these properties and $[JC : H]$ is a bounded operator. Hence, by Theorem 6.1.4 in [1], a unique mild solution of (5.6) exists for all t such that $x(t)$ and $z(t)$ are bounded. In Lemma 5 below, we will show that if Δ satisfies the Popov inequality then x and z cannot finitely escape. Hence, a unique mild solution exists for all $t \geq 0$. The case considered in [34] and [§IV.2,39] is a special case of this formulation. The interconnected system is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + b\phi(\sigma(t)) \\ \frac{d\sigma(t)}{dt} &= \langle c, x(t)\rangle - \phi(\sigma(t))\rho \quad . \end{aligned} \tag{5.7}$$

In our framework, this corresponds to $\ell = 1$, $B = -b$, $C = \langle c, \cdot\rangle$, $D = \rho$, $F = 0$, $G = 1$, $H = 1$, $J = 0$ and ϕ time invariant. Since $\|JD\| = 0$, the assumption on ϕ_1 can be directly placed on ϕ. We shall discuss this example again later in this section. Another special case is when Δ is also a linear time invariant system modeled by an evolution equation. In this case, ϕ is just the identity map and a unique mild solution of (5.6) exists.

The following lemma is needed to show the differentiability of the energy function $V(x(t))$ along the solution.

Lemma 5. Consider the interconnected system in Fig. 1. Assume $D > 0$. If Δ satisfies the Popov inequality, then x does not finitely escape, $x \in L_{2_e}(\mathbf{X})$, $u \in L_{2_e}(\mathbf{R}^m)$ and there exist positive constants η_1 and η_2 such that

$$\|u\|_t \leq \eta_1 + \eta_2 \|x\|_t \quad . \tag{5.8}$$

If Δ satisfies the exponential Popov's inequality, then x does not finitely escape, $e^{\sigma t}x(t) \in L_{2_e}$, $e^{\sigma t}u(t) \in L_{2_e}$, and there exist positive constants σ, η_1 and η_2 such that

$$\|e^{\sigma s}u(s)\|_t \leq \eta_1 + \eta_2 \|e^{\sigma s}x(s)\|_t \quad . \tag{5.9}$$

Proof: The proof is given in Appendix X .

■

If Δ is given by (5.1) and F generates a *bounded* C_o–semigroup (i.e., the semigroup is bounded uniformly in t), then under the conditions in Lemma 5, z does not finitely escape either. Hence, condition (5.5) , the Popov's inequality on Δ and $D > 0$ guarantees the existence of a unique mild solution for all $t \geq 0$.

We can now state and prove the main result of this section.

Theorem 4. Given an interconnected system as in Fig. 1, let the forward system $\mathcal{T}$ be an exponentially stable system given by (1.1) . If $\nu(\mathcal{T}) < 0$ and Δ satisfies the Popov inequality, then $x(t) \to 0$ as $t \to \infty$.

Proof: Let $V(x) = \langle Px, x\rangle$, where P is given by the solution of the Lur'e equation (2.1) associated with $\mathcal{T}$. From Lemma 5, $u \in L_{2_e}$. Hence, by Lemma 1, $V(x(t))$ is differentiable along the mild solution (2.2) and $\dot{V}(x(t))$ is given by (some algebra is skipped since it is identical to that in the proof of Theorem 3):

$$\dot{V}(t, x(t)) \leq -\epsilon \|x(t)\|^2 + 2\langle y(t), u(t)\rangle \quad .$$

By integrating both sides of the inequality from 0 to ∞, we have

$$\int_0^\infty \|x(s)\|^2\, ds \le \frac{1}{\epsilon}\left(V(x_o) + 2\limsup_{t\to\infty}\int_0^t \langle y(s), u(s)\rangle\, ds\right)$$
$$\le \frac{1}{\epsilon}\left(V(x_o) + 2\xi\right) < \infty \quad .$$

Hence, $x \in L_2([0,\infty);\mathbf{X})$. From (5.8), this implies that $u \in L_2([0,\infty);\mathbf{R}^m)$. From Lemma 2, it follows that $x(t) \to 0$ as $t \to \infty$ for all $x_o \in \mathbf{X}$.

∎

Under additional assumptions, the asymptotic stability result in Theorem 4 can be strengthened to exponential stability.

Corollary 5. In Theorem 4, If, in addition, Δ satisfies the exponential Popov inequality, then $x(t) \to 0$ exponentially as $t \to \infty$.

Proof: Define

$$V_1(t,x) = e^{2\sigma t}V(x) \quad ,$$

where $V(x)$ is as defined in the proof of Theorem 4. Since $V(x(t))$ is differentiable along the mild solution of $\mathcal{T}$, so is $V_1(t,x(t))$ and the derivative is given by

$$\dot{V}_1(t,x(t)) \le 2\sigma e^{2\sigma t}\langle Px(t), x(t)\rangle + e^{2\sigma t}\left(-\epsilon\|x(t)\|^2 + 2\langle y(t), u(t)\rangle\right)$$
$$\le -e^{2\sigma t}(\epsilon - 2\sigma\|P\|)\|x(t)\|^2 + 2e^{2\sigma t}\langle y(t), u(t)\rangle \quad .$$

By integrating both sides and using the exponential Popov's inequality assumption, we have $e^{\sigma t}x(t) \in L_2([0,\infty);\mathbf{X})$ for all $\sigma \in (0, \frac{1}{2}\epsilon\|P\|^{-1})$. By Lemma 5, this implies that $e^{\sigma t}u(t) \in L_2([0,\infty);\mathbf{R}^m)$. It follows from Lemma 2 that $x(t) \to 0$ exponentially.

∎

Remarks:

7. Theorem 4 and Corollary 5 only state that $x(t) \to 0$ as $t \to \infty$ and the feedback system is L_2 stable, but not the convergence of the internal signals in the feedback system. If the feedback system is given by (5.1) in which F generates an exponentially stable C_o–semigroup, then the feedback system is also internally stable.

8. If Δ is an exponentially stable linear time invariant system that can be represented as (1.1), then by Proposition 1, the condition on Δ in Theorem 4 and Corollary 5 can be replaced by positive realness and strict positive realness, respectively.

∎

The same loop transformation technique used in Section 4 can be applied here, also. We first state the result relating to the transformation in Fig. 2.

Corollary 6. Consider the interconnected system in Fig. 1. Assume $\mathcal{T}$ is exponentially stable. If

$$\nu(T) < \frac{1}{\beta} \quad, \beta > 0, \tag{5.10}$$

and Δ satisfies

$$\langle y, \Delta(y)\rangle_t - \frac{1}{\beta}\|\Delta(y)\|_t^2 \ge -\xi \tag{5.11}$$

for all $t \in [0,\infty)$ and $y \in \mathbf{R}^m$ and some $\xi \ge 0$, then $x(t) \to 0$ as $t \to \infty$.

Proof: The transformed forward system, $\mathcal{T} + \frac{1}{\beta} \cdot I$ is strictly positive real by assumption. Let $\tilde{y}$ be the input into the transformed feedback system. Then

$$\tilde{y} = y - \frac{1}{\beta}\Delta(t, y) \quad .$$

The L_2-innerproduct between the input and output of the transformed feedback system is

$$\langle \tilde{y}, \Delta(y) \rangle_t = \langle y, \Delta(y) \rangle_t - \frac{1}{\beta} \|\Delta(y)\|_t^2 \quad ,$$

which, by assumption, is uniformly bounded below. Hence, asymptotic convergence of $x(t)$ to zero follows from 4.

■

If the feedback system is given by (5.1) and the transformed feedback system satisfies the Popov's inequality, then all the internal signals are finitely bounded (by Lemma 5). Therefore, (5.5) implies a unique mild solution exists for all $t \geq 0$.

Similar to Corollary 3, the stability result related to the transformation in Fig. 3 can be easily derived.

Corollary 7. Consider the interconnected system in Fig. 1. Define

$$\tilde{\mathcal{T}} = (I - \alpha \mathcal{T})^{-1} \mathcal{T} \quad , \tag{5.12}$$

where $\frac{1}{\alpha} \notin \sigma(D)$. Let the transfer function representation of $\tilde{\mathcal{T}}$ be $\tilde{T}(s)$. If $\tilde{\mathcal{T}}$ is exponentially stable and

$$\nu(\tilde{T}) < \frac{1}{\beta} \quad , \beta > 0, \tag{5.13}$$

and Δ satisfies

$$\langle (\Delta(y) + \alpha y), (\Delta(y) - (\beta - \alpha) y) \rangle_t \leq \xi \tag{5.14}$$

for all $t \in [0, \infty)$ and $y \in \mathbf{R}^m$ and some $\xi \geq 0$, then $x(t) \to 0$ as $t \to \infty$.

Proof: In Fig. 3, the transformed forward system is $\tilde{\mathcal{T}}$ and the transformed feedback system is $\Delta(y) + \alpha y$. Since the transformed feedback system satisfies the Popov's inequality, by Lemma 5, $u \in L_{2_e}$ and $x \in L_{2_e}$, which implies that the input into the transformed forward system is in L_{2_e}. Corollary 6 can now be used to complete the proof.

■

Remarks:

9. In Theorem 4 and Corollaries 5—7, if Δ is a time-invariant system, the resulting stability of the interconnected system is uniform with respect to the initial time.

10. If $\beta = 2\alpha$, then the class of Δ that satisfies (5.14) is equivalent to an L_{2_t}-norm upperbound on Δ for all $t \in [0, \infty)$.

11. Results in this section can be directly applied to the type of systems addressed in [34]. Consider the system given by (5.7), where A generates an exponentially C_o-semigroup and ϕ is a nonlinear, locally

Lipschitz function that satisfies $r\phi(r) \geq 0$ and $\phi(0) = 0$. The wellposedness of this system has been shown in the beginning of this section. We want to show that more general stability conditions can be obtained by using the results here. This system can be represented in terms of the block diagram of Fig. 4. The forward system has the transfer function

$$T(s) = \rho - c(sI - A)^{-1}b \quad .$$

The feedback system consists of an integrator and ϕ, which is time invariant. The L_{2_t}-inner-product between the input and output of the feedback system is computed as

$$\int_0^t \phi(\sigma(\tau))\frac{d\sigma(\tau)}{d\tau}\,d\tau = \int_{\sigma(0)}^{\sigma(t)} \phi(\sigma)\,d\sigma \quad .$$

Define $\Phi(\sigma) = \int_0^\sigma \phi(\xi)\,d\xi$. Since ϕ is a first–third quadrant function, $\Phi(\sigma) \geq 0$ for all $\sigma \in \mathbf{R}$. Hence,

$$\int_{\sigma(0)}^{\sigma(t)} \phi(\sigma)\,d\sigma = \Phi(\sigma(t)) - \Phi(\sigma(0)) \geq -\Phi(\sigma(0))$$

which implies that the feedback system satisfies the Popov inequality. By Theorem 4, if $\nu(T) < 0$, then the system described by (5.7) is uniformly asymptotically stable. This result is more general than that in [34] in which ϕ is required to satisfy an additional condition $\lim_{|s| \to \infty} \int_0^s \phi(\sigma)\,d\sigma = \infty$ for the closed loop asymptotic stability. Furthermore, Corollaries 6—7 can be used for more general classes of forward and feedback systems. For example, the reactor type equation in Eq. (1.3) of [34] is a case that the stability condition in [34] is not directly applicable (since $\rho = 0$) but can still be considered within the framework here.

12. Absolute stability theorem, Theorem 3, can be considered as a special case of Corollary 5. If Δ is non-negative for each t, it satisfies the exponential Popov inequality. Then, by Corollary 5, $x(t) \to 0$ exponentially as $t \to \infty$.

6. Robustness Analysis for Unstructured Uncertainties

Results in sections 4–5 can be interpreted in a robustness analysis context. Restate the result in Corollary 7 as:

For all β that satisfies

$$\beta < \frac{1}{\nu((I - \alpha T)^{-1}T)} \quad , \tag{6.1}$$

a class of Δ that preserves stability is given by

$$\Sigma_1(\alpha,\beta) = \{\Delta : \langle(\Delta(t,y) + \alpha y), (\Delta(t,y) - (\beta - \alpha)y)\rangle_t \leq \xi \quad \text{for some } \xi,\ \text{all } t \text{ and all } y \in L_{2_t}\} \quad . \tag{6.2}$$

We only consider the uncertainty class as in (6.2) ; the absolute stability case (where inner product is taken in $\mathbf{R}^m$) follows in exactly the same way. In this section, we will analyze conditions (6.1) and (6.2) in greater detail; specifically, we want to restate them in terms of conditions directly on T and Δ, respectively.

First consider (6.2) . In the SISO case ($m = 1$), this condition has a natural interpretation of sector-boundedness: Δ lies between two lines: $-\alpha y$ and $(\beta - \alpha)y$. We call (6.2) a general sector-boundedness condition on Δ for $m > 1$; however, the interpretation is less clear. We seek to relate Σ_1 to the following set that has greater practical appeal (a norm upperbound and an inner-product lower bound):

$$\begin{aligned}\Sigma_2(\alpha,\beta) = \{\Delta : \|\Delta(t,y)\|_t \le (\beta-\alpha)\,\|y\|_t + \gamma_1 \quad , \quad \langle \Delta(t,y), y\rangle_t \ge -\alpha\,\|y\|_t^2 - \gamma_2, \\ \text{for some } \gamma_1,\gamma_2 > 0 \text{ all } t \text{ and all } y \in L_{2_t}\} \quad .\end{aligned} \tag{6.3}$$

It is easy to see that $\Sigma_1(\alpha,\beta) \subset \Sigma_2(\alpha,\beta)$. However, the reverse inclusion is of greater interest since we would like to have the stability condition (6.1) to directly provide an acceptable class of Δ given by Σ_2. Unfortunately, the reverse inclusion is not true in general, but it is true in the following two special cases.

1. $\beta = 2\alpha$.
2. In $\Sigma_1(\alpha,\beta)$ and $\Sigma_2(\alpha,\beta)$, replace $\|\cdot\|_t$ and $\langle\cdot,\cdot\rangle_t$ by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, γ_i's by zero. Δ in Σ_1 is assumed to be linear and symmetric.

Case 1 follows from algebra. The set $\Sigma_2(\alpha, 2\alpha)$ then reduces to a single norm upperbound of Δ by α.

In case 2, there exists a symmetric matrix $\bar\Delta$ for each t such that

$$\Delta(t,y) = \bar\Delta(t)y \quad .$$

Suppose $\Delta \in \Sigma_2(\alpha,\beta)$, i.e.,

$$\begin{aligned}\|\bar\Delta(t)\| &\le \beta - \alpha \\ \mu_{\min}\left(\bar\Delta(t)\right) &\ge -\alpha \quad ,\end{aligned} \tag{6.4}$$

for all $t \ge 0$. Since $\bar\Delta(t) + \alpha\cdot I$ is symmetric non-negative definite, there exists a factorization:

$$\bar\Delta(t) + \alpha\cdot I = M^T M \quad .$$

Now,

$$\begin{aligned}&\|(\bar\Delta+\alpha\cdot I)y\|^2 - \beta y^T(\bar\Delta+\alpha\cdot I)y \\ =&\left(\frac{\|(\bar\Delta+\alpha\cdot I)y\|^2}{y^T(\bar\Delta+\alpha\cdot I)y} - \beta\right)\cdot y^T(\bar\Delta+\alpha\cdot I)y \\ \le&\left(\frac{\|M^TMy\|^2}{\|My\|^2} - \beta\right)\cdot y^T(\bar\Delta+\alpha\cdot I)y \\ \le&\left(\|M\|^2 - \beta\right)\cdot y^T(\bar\Delta+\alpha\cdot I)y \\ \le&\left(\|\bar\Delta+\alpha\| - \beta\right)\cdot y^T(\bar\Delta+\alpha\cdot I)y \\ \le&\left(\|\bar\Delta\| - (\beta-\alpha)\right)\cdot y^T(\bar\Delta+\alpha\cdot I)y \\ \le&0 \quad .\end{aligned}$$

Hence, $\Delta \in \Sigma_1(\alpha,\beta)$.

The inclusion $\Sigma_2(\alpha,\beta) \subset \Sigma_1(\alpha,\beta)$ is not true in general, witness antisymmetric $\bar\Delta + \alpha\cdot I$. The following result shows a connection between the two sets. If

$$(\beta_1-\alpha_1)^2 + (\beta-2\alpha)\alpha_1 - \alpha(\beta-\alpha) \le 0 \quad , \tag{6.5}$$

then

$$\Sigma_2(\alpha_1,\beta_1) \subset \Sigma_1(\alpha,\beta) \quad .$$

Given (α, β), the allowable region of (α_1, β_1) is shown in the shaded area in Fig. 5. This figure is useful in finding an acceptable class of $\Delta \in \Sigma_2(\alpha_1, \beta_1)$, given a pair (α, β) from Corollary 4 (a graphic method will be discussed later in this section). Given (α_1, β_1), the allowable (α, β) is shown in the shaded area in Fig. 6. This figure can be used to transform uncertainty specification from $\Sigma_2(\alpha_1, \beta_1)$ to $\Sigma_1(\alpha, \beta)$.

For the rest of this section, we focus on (6.1). We assume $\beta - \alpha \geq \alpha$; otherwise, the lowerbound in (6.3) is redundant. A more convenient sufficient condition for (6.1) is stated below.

Proposition 5. If T is exponentially stable and there exists $\eta > 0$ such that

$$1 + (\beta - 2\alpha)[\mathrm{RF}(T)](\omega) - \alpha(\beta - \alpha)\,\|T(j\omega)\|^2 \geq \eta \quad , \tag{6.6}$$

for all $\omega \in \mathbf{R}$, then $\nu(\widetilde{T}) < \frac{1}{\beta}$ where $\widetilde{T}$ is given by (4.9).

Proof: If (6.6) holds, then for all $v \in \mathbf{C}^m$

$$(\beta - 2\alpha)\mathrm{Re}\, v^* T v + \|v\|^2 - \alpha(\beta - \alpha)\,\|T^* v\|^2 \geq \eta\,\|v\|^2$$

which can be written as

$$\mathrm{Re}\, v^*((\beta - \alpha)T + I)(I - \alpha T^*)v \geq \eta\,\|v\|^2 \quad . \tag{6.7}$$

For each $\omega \in \mathbf{R}$ and $z \in \mathbf{C}^m$, define

$$v_\omega = (I - \alpha T^*(j\omega))^{-1} z \quad . \tag{6.8}$$

By Theorem 5.7.1 in [56], we have

$$\|(I - \alpha T^*(j\omega))^{-1}\| \geq \frac{1}{1 + \alpha\,\|T\|_{H_\infty}} \quad . \tag{6.9}$$

Hence, there exists a $\delta > 0$, independent of ω, such that

$$\|v_\omega\|^2 \geq \delta\,\|z\|^2 \quad . \tag{6.10}$$

Substitute v_ω for v in (6.7) and use (6.10), we have

$$\mathrm{Re}\, z^*(I - \alpha T(j\omega))^{-1}\left((\frac{\beta - \alpha}{\beta})T(j\omega) + \frac{1}{\beta} \cdot I\right) z \geq \xi\,\|z\|^2$$

for some $\xi > 0$ and all $z \in \mathbf{C}^m$. This can be written as

$$\mathrm{Re}\, z^*((I - \alpha T(j\omega))^{-1}T(j\omega) + \frac{1}{\beta})z \geq \xi\,\|z\|^2$$

which is equivalent to $\nu(\widetilde{T}) < \frac{1}{\beta}$.

■

Remarks:

13. Note that (6.7) has similar form as Δ in Σ_1. Therefore, we say T is sector bounded with a general "sector" $(-\frac{1}{(\beta-\alpha)}, \frac{1}{\alpha})$.

14. A sufficient condition of (6.6) is

$$\alpha(\beta-\alpha)\,\|T\|^2_{H_\infty} < 1-(\beta-2\alpha)\nu(T) \quad . \tag{6.11}$$

If $\beta = 2\alpha$, then (6.6) becomes

$$\beta-\alpha = \alpha < \frac{1}{\|T\|_{H_\infty}} \quad . \tag{6.12}$$

Recall that when $\beta = 2\alpha$ the class of Δ that preserves stability is characterized by the small gain conditions (see Remarks 5 and 10). Hence, we have obtained the following small gain stability criterion [40]:

$$\|\Delta\| \le \frac{1}{\|T\|_{H_\infty}} \quad \text{or} \quad \|\Delta\|_t \le \frac{\|y\|_t}{\|T\|_{H_\infty}} + \gamma \tag{6.13}$$

for some $\gamma > 0$.

If $\alpha = 0$, then (6.6) becomes

$$\beta < \frac{1}{\nu(T)} \quad . \tag{6.14}$$

This case is a restatement of Corollary 4 and 7.

If $\nu(T) = 0$ (i.e., T is positive real), then (6.13) becomes

$$\alpha(\beta-\alpha) < \frac{1}{\|T\|^2_{H_\infty}} \quad . \tag{6.15}$$

This condition demonstrates the trade–off between the upperbound and lowerbound of Δ when T is positive real ($\beta-\alpha$ and α are interpreted as generalized upperbound and lowerbound of Δ, respectively; see Remark 5).

15. Another special case of interest is when Δ is a constant linear scalar. Write Δ as $\tilde{\Delta} y$. Then an acceptable class of $\tilde{\Delta}$ is

$$-\frac{1}{\nu(-T)} < \tilde{\Delta} < \frac{1}{\nu(T)} \quad . \tag{6.16}$$

■

Given a system T, condition (6.6) can be checked by a single graphic test. Write (6.6) as

$$\mathrm{sgn}(\alpha)\left[-\left(\mathrm{RF}(T)(\omega)-\frac{1}{\alpha}\right)\left(\mathrm{RF}(T)(\omega)+\frac{1}{\beta-\alpha}\right)+\mathrm{RF}^2(T)(\omega)-\|T(j\omega)\|^2\right] > \eta \quad . \tag{6.17}$$

Define

$$z(\omega)^2 = (\,\|T(j\omega)\|^2 - \mathrm{RF}^2(T)(\omega))$$

where the sign of z is chosen the same as the sign of ω. If $z(\omega)$ is plotted versus $\mathrm{RF}(T)(\omega)$, then (6.17) is equivalent to the plot staying within the circle symmetric about the $\mathrm{RF}(T)(\omega)$ axis with end points $-\frac{1}{\beta-\alpha}$ and $\frac{1}{\alpha}$, for $\alpha \ge 0$, and staying out of the circle with end points $-\frac{1}{\beta+|\alpha|}$ and $-\frac{1}{|\alpha|}$, for $\alpha \le 0$. The forbidden region in each of the two cases, $\alpha \ge 0$ and $\alpha \le 0$, is illustrated in Fig. 7 .

The z vs. $\mathrm{RF}(T)$ plot is similar to a Nyquist plot except T can be a matrix (if T is a scalar transfer function, then this plot is just the Nyquist plot with perhaps a flip about the ReT axis). The graphic test itself is similar to the circle criterion [24,40]. Since it reduces exactly to the circle criterion in the lumped parameter, single–input/single–output case, this test can be considered as a generalization of the circle criterion. Note that α and β need not be selected a priori. Once the Nyquist–like plot is given, α and β can be chosen so that the corresponding circles either enclose or stay away from the plot, depending on the sign of α. Of course, α is additionally constrained to feedback stabilize $\mathcal{T}$.

This version of the circle criterion for evolution equations is similar to the previous generalizations in [35, 36]. The setting here is slightly different in that the input and output spaces are assumed to be finite dimensional. As a consequence, we are able to obtain the graphic test involving a single Nyquist–like plot.

Remarks:

16. To apply this technique in robustness analysis, we suggest the following procedure. If $m = 1$, use (6.16). If $m > 1$, choose α and $\beta - \alpha$ from the generalized circle criterion (by using fig. 7) with the constraint that $\alpha < \frac{1}{\nu(-\mathcal{T})}$. Then interpret the acceptable class of Δ by either $\Sigma_1(\alpha,\beta)$ or $\Sigma_2(\alpha_1,\beta_1)$ with (α_1,β_1) satisfies (6.5).

■

7. Robustness Analysis for Structured Uncertainties

When additional structure is known in Δ, stronger stability results can be obtained. In this section, we assume Δ is diagonal and time invariant. By increasing the assumption on the elements of Δ, from a general time invariant sector bounded nonlinearity to monotone nonlinearity to linear scalars, progressively improved robustness margins can be obtained.

To proceed with the discussion, we need to introduce the multiplier technique for robustness analysis. At the present, it is necessary to restrict $\mathcal{T}$ to finite dimensional systems due to the boundedness requirement on the input and output operators (B and C). The generalization to evolution systems is included in the research thrust to extend the passivity approach in this report to unbounded input and output operators.

Consider a scalar, non–negative Δ in Fig. 1. A multiplier, z, is an operator that changes the passivity of $z\mathcal{T}$ from that of $\mathcal{T}$, but does not change the passivity of $\Delta \cdot \frac{1}{z}$ from that of Δ. We can use this technique together with the loop transformation introduced in Sections 4 and 5. Consider the system in Fig. 1, assuming both systems are scalar. After feedback of α and feedforward of $\frac{1}{\beta}$, the forward system becomes $\frac{\mathcal{T}}{I+\alpha\mathcal{T}} + \frac{1}{\beta}$. Call the feedback system $\tilde{\Delta}$, with input $\tilde{y}$ and output w, after the corresponding loop transformations having been performed on Δ. If there exists a multiplier z for the transformed interconnected system such that

$$\nu\left(z \cdot \left(\frac{\mathcal{T}}{I+\alpha\mathcal{T}} + \frac{1}{\beta}\right)\right) \leq 0 \quad ,$$

then $x(t) \longrightarrow 0$ as $t \longrightarrow \infty$ so long as

$$\tilde{y}w \geq 0 \quad . \tag{7.1}$$

Now, $y = \frac{w}{\beta} + \tilde{y}$. Therefore, $\tilde{y} = y - \frac{w}{\beta}$. The condition for stability (7.1) becomes

$$
\begin{aligned}
&(y - \frac{w}{\beta})w \geq 0 \\
\Leftrightarrow & yw \geq \frac{w^2}{\beta} \\
\Leftrightarrow & w^2 \leq \beta yw \\
\Leftrightarrow & 0 \leq (\Delta(y) + \alpha y)^2 \leq \beta y(\Delta(y) + \alpha y) \\
\Leftrightarrow & 0 \leq (\frac{\Delta(y)}{y} + \alpha) \leq \beta \\
\Leftrightarrow & -\alpha \leq \frac{\Delta(y)}{y} \leq \beta - \alpha \quad .
\end{aligned}
$$

This then motivates the following optimization problem for finding the "best" multiplier, given a lower bound α for Δ:

Find z from a specified class of multipliers which maximizes β that satisfies

$$\nu\left(z \cdot (\frac{\mathcal{T}}{I + \alpha\mathcal{T}} + \frac{1}{\beta})\right) \leq 0 \quad . \tag{7.2}$$

In the rest of this section, we will discuss results related to special classes of multipliers and their applications to diagonal uncertainties.

If Δ is a scalar, time invariant nonlinearity, the following is a legitimate multiplier

$$z_1(j\omega) = 1 + qj\omega \quad \text{for any } q \geq 0 \quad . \tag{7.3}$$

If Δ is a scalar monotone nonlinearity, in addition to z_1, another valid multiplier is

$$z_2(j\omega) = 1 + qj \quad \text{for any } q \in \mathbf{R} \quad . \tag{7.4}$$

When Δ is diagonal and linear, the transformation $D\Delta D^{-1} = \Delta$ does not affect the feedback system but changes the passivity of the forward system, $D^{-1}\mathcal{T}D$. This technique is called D-scaling (since it represents a rescaling of the input and output of $\mathcal{T}$) and has been used extensively in the computation of the μ measure [12].

First consider the multiplier z_1. If Δ is a scalar time invariant, first-third quadrant nonlinearity, then the L_{2_t} inner product between the input and output of $\Delta \cdot \frac{1}{z_1}$ is

$$\left\langle (\Delta \cdot \frac{1}{z_1})(y), y \right\rangle_t = \langle \Delta(y), y \rangle_t + \langle \Delta(y), \dot{y} \rangle_t \quad . \tag{7.5}$$

By assumption, the first term in (7.5) is non-negative. The second term can be written as

$$\int_{y(0)}^{y(t)} \Delta(y)\, dy \quad .$$

,et $Y = \int_0^y \Delta(\xi)\,d\xi$. Then

$$\int_{y(0)}^{y(t)} \Delta(y)\,dy = Y(y(t)) - Y(y(0)) \geq -Y(y(0)) \quad .$$

Hence, $\Delta \cdot \frac{1}{z_1}$ satisfies the Popov inequality. Assume $\mathcal{T}$ is strictly proper ($D = 0$). Then $z_1\mathcal{T}$ is proper with particular realization $(A, B, C + qCA, qCB)$. The difficulty for infinite dimensional generalization is clear: y may not be differentiable and Theorem 2 does not allow unbounded input and output operators. Since $\mathcal{T}$ ̇ȷ assumed finite dimensional,

$$\nu(z_1\mathcal{T}) < 0 \quad , \tag{7.6}$$

implies that the state of $z_1\mathcal{T}$ converges to zero asymptotically, by Corollary 6 . Since z_1 only affects the utput map of $\mathcal{T}$, the state of $\mathcal{T}$ converges to zero, also. If the upperbound of Δ is β, then the stability condition (7.6) becomes $\nu(z_1(\mathcal{T} + \frac{1}{\beta})) < 0$ which is equivalent to

$$\nu(z_1\mathcal{T}) < \frac{1}{\beta} \quad . \tag{7.7}$$

The stability condition obtained by using the multiplier z_1 is called the Popov criterion and has a graphical ıterpretation [40,54,25].

Condition (7.7) can be posed as an optimization problem: find q in z_1 that maximizes β. An equivalent roblem is to find q that minimizes $J(q) = \nu(z_1\mathcal{T})$. We now show that $J(q)$ is globally convex in q, therefore, the optimization problem can be efficiently solved by using, for example, a line search technique. The convexity of $J(q)$ is shown below:

$$\begin{aligned}
&J(\alpha q_1 + (1-\alpha)q_2) \\
=&\nu\left((1 + (\alpha q_1 + (1-\alpha)q_2)j\omega)\mathcal{T}(j\omega)\right) \\
=&\nu\left(\alpha(1 + q_1 j\omega)\mathcal{T}(j\omega) + (1-\alpha)(1 + q_2 j\omega)\mathcal{T}(j\omega)\right) \\
\leq&\alpha\nu((1 + q_1 j\omega)\mathcal{T}(j\omega)) + (1-\alpha)\nu((1 + q_2 j\omega)\mathcal{T}(j\omega)) \\
&\quad \text{(by parts 4 and 5 of Fact 2)} \\
=&\alpha J(q_1) + (1-\alpha)J(q_2) \quad .
\end{aligned}$$

If Δ is a scalar monotone nonlinearity, the multiplier z_2 can be used in addition to z_1. This fact was proved in [p.166 in 25] and [57] and was shown leading to the off–axis circle graphic test. If Δ is non-negative ith upperbound β, then a condition for stability based on z_2 is $\nu(z_2(\mathcal{T} + \frac{1}{\beta})) < 0$ which is equivalent to

$$\nu(z_2\mathcal{T}) < \frac{1}{\beta} \quad . \tag{7.8}$$

gain an optimization problem can be posed to find q that minimizes $J(q) = \nu(z_2\mathcal{T})$. With analysis identical to the z_1 case before (with $\omega = 1$), it is easy show that $J(q)$ is globally convex in q; therefore, the optimization ·oblem can again be efficiently solved.

We now consider Δ with scalar diagonal elements, Δ_i, $i = 1, \ldots, m$. If each Δ_i is a non-negative ›nlinearity (possibly non-monotone), then a straightforward generalization of the scalar result leads to the llowing bound on $\beta \triangleq \max_i |\Delta_i|$:

$$\beta < \min\left\{\frac{1}{\nu(\mathcal{T}Z_1)}, \frac{1}{\nu(Z_1\mathcal{T})}\right\} \quad . \tag{7.9}$$

where $Z_1 = \operatorname{diag}\{1+q_1 j\omega, \ldots, 1+q_m j\omega\}$. The optimization problems of finding $q = \operatorname{col}\{q_1, \ldots, q_m\}$ to minimize $J(q) = \nu(TZ_1)$ or $J(q) = \nu(Z_1 T)$ are again globally convex in q.

If each Δ_i is a non-negative monotone nonlinearity, then the inverse of a upperbound on $\max_i |\Delta_i|$ can be found from the minimization of the indices $J(q) = \nu(TZ_i)$ or $J(q) = \nu(Z_i T)$ for $i = 1,2$ and $Z_2 = \operatorname{diag}\{1+q_1 j, \ldots, 1+q_m j\}$, which are all globally convex in q.

In applying the above results, a lowerbound, $L = \operatorname{diag}\{-\alpha_1, \ldots, -\alpha_m\}$, should be subtracted from each Δ_i. Then upperbounds for $\Delta_i + \alpha_i y$ can be found by using the multiplier technique with T replaced by $(I+TL)^{-1}T$.

When each Δ_i is, in addition, linear, we obtain a superior stability criterion. Write each Δ_i as $\tilde{\Delta}_i y$. Consider 2^m different cases of possible variations of $\tilde{\Delta}$, depending on the signs of each $\tilde{\Delta}_i$. Let $S = \operatorname{diag}\{s_1, \ldots, s_m\}$, where s_i is either $+1$ or -1. Clearly, there are 2^m possible S. By combining with the multiplier technique before, we have the following stability condition for each quadrant of the parameter space:

$$s_i \tilde{\Delta}_i \geq 0 \tag{7.10a}$$

$$\|\tilde{\Delta}\| = \max_i |\tilde{\Delta}_i| < \max_{k=1,2} \max \left\{ \frac{1}{\inf_q \nu(STZ_k(q))}, \frac{1}{\inf_q \nu(Z_k(q)ST)} \right\} \quad , \tag{7.10b}$$

where $s_i = \pm 1$.

Since Δ is linear, we can further incorporate the D–scaling into (7.10) :

$$s_i \tilde{\Delta}_i \geq 0 \tag{7.11a}$$

$$\|\tilde{\Delta}\| = \max_i |\Delta_i| < \sup_D \max_{k=1,2} \max \left\{ \frac{1}{\inf_q \nu(DSTD^{-1}Z_k(q))}, \frac{1}{\inf_q \nu(Z_k(q)DSTD^{-1})} \right\} \quad , \tag{7.11b}$$

where $D = \operatorname{diag}\{d_1, \ldots, d_m\}$, $d_i \in \mathbf{R}$.

It can be easily shown that the minimum of the bounds in (7.11) is a less conservative upperbound for the μ measure than a common choice in the literature: $\inf_D \|DTD^{-1}\|_{H_\infty}$, and, furthermore, it is a bound for the real parameter variations (i.e., $\tilde{\Delta}_i \in \mathbf{R}$).

The stability bounds in (7.10) and (7.11) measure "directional" robustness. The added information over a single scalar measure such as the μ measure may be useful in the following context. If Δ can be modified, then directional robustness can point to a set of parameters that possesses better robustness property. Frequently, the true plant is a linearized version of some nonlinear system. If Δ corresponds to uncertain plant parameters, then the results here can point to a more robust operating point. If Δ corresponds to uncertain gains in the controller, then direction robustness is again useful in selecting a robust nominal value.

We have only consider two specific choices of multipliers so far in this section. This is due to their simple forms and the global convexity in the corresponding optimization problem. In general, there exists a large class of multiplier for monotone nonlinearities [58, 59]:

$$z(j\omega) = 1 - \int_{-\infty}^{\infty} z_1(t) e^{-j\omega t}\, dt \tag{7.12}$$

$$z_1(t) > 0 \quad \text{for all } t \in \mathbf{R} \tag{7.13}$$

$$\int_{-\infty}^{\infty} z_1(t)\, dt < 1 \quad . \tag{7.14}$$

For the rest of this section, we will present some preliminary thoughts on finding the optimal multiplier within this class. Suppose Δ is scalar and non-negative. Then $\nu[(\mathcal{T}+\frac{1}{\beta})z] < 0$ is a sufficient condition for stability. A numerical algorithm was proposed in [60] to find the largest β. To reduce the problem to an optimization involving finite number of parameters, t and ω are discretized over uniform and finite grids. The algorithm then finds the optimal multiplier for repeated guesses of β until $\nu[(\mathcal{T}+\frac{1}{\beta})z] < 0$. We propose a different procedure. Write the optimization problem in a slightly different form. By straightforward algebra, it is easy to show that

$$\nu[(\mathcal{T}+\frac{1}{\beta})z] < 0 \Leftrightarrow \beta < g\left[\inf_{\omega} \frac{\operatorname{Re} z(j\omega)}{\operatorname{Re}(T(j\omega)z(j\omega))}\right] \tag{7.15}$$

where

$$g(x) \stackrel{\Delta}{=} \begin{cases} \infty & \text{if } x \geq 0 \\ |x| & \text{if } x < 0 \end{cases} .$$

Since $z_1(t) \geq 0$, there exists $z_2 \in L_2(-\infty,\infty)$ such that

$$\int_{-\infty}^{\infty} z_1(t) e^{-j\omega t}\, dt = \langle z_2 h, z_2 \rangle \quad , \tag{7.16}$$

where the inner product is the complex $L_2(-\infty,\infty)$ inner product over the variable t, and h is given by

$$h(j\omega) = e^{j\omega t} \stackrel{\Delta}{=} h_r(\omega) + j h_i(\omega)$$

$$h_r(\omega) = \cos\omega t \qquad h_i(\omega) = \sin\omega t \quad .$$

Let T_r and T_i be the real and imaginary part of $\mathcal{T}$. Then the problem of finding the optimal multiplier can be written as

$$\beta^* = \sup_{\substack{z_2 \in L_2(-\infty,\infty) \\ \|z_2\|_2 \leq 1}} \inf_{\omega \in \mathbf{R}} g\left[\frac{1-\langle h_r z_2, z_2\rangle}{T_r - T_r\langle h_r z_2, z_2\rangle + T_i \langle h_i z_2, z_2\rangle}\right] \quad . \tag{7.17}$$

This is an infinite–dimensional optimization problem since z_2 is in a unit ball in $L_2(-\infty,\infty)$. Let $\{e_i\}_{i=1}^{\infty}$ be a basis of $L_2(-\infty,\infty)$ (such basis exists since $L_2(-\infty,\infty)$ is separable). Approximate z_2 by

$$z_2 = \sum_{i=1}^{N} a_i e_i = Ea \quad , \tag{7.18}$$

where $E = \{e_1,\ldots,e_N\}$ and $a = \operatorname{col}\{a_1,\ldots,a_N\}$. Let H_r be the symmetric matrix with the (i,j)th element $\langle h_r e_i, e_j\rangle$ and H_i be the matrix with $\langle h_i e_i, e_j\rangle$. Then the optimal multiplier problem can be approximated by the following finite dimensional problem:

$$\beta_N = \sup_{\|a\|\leq 1} \inf_{\omega \in \mathbf{R}} g\left[\frac{1-a^T H_r a}{T_r - T_r a^T H_r a + T_i a^T H_i a}\right] \quad .$$

Many constrained nonlinear optimization algorithms can be used for this problem (e.g.,[61]), but issues such as the rate of convergence and numerical stability remain to be explored. β_N is clearly nondecreasing in N and bounded above by β^*. If β^* is bounded, then β_N will converge, but whether it will converge to β^* remains to be investigated.

Convergence of nonlinear numerical optimization algorithms depend heavily on the accuracy of the initial guess. Therefore, as N is increased, optimal a from the previous N, padded with zeros, should be used as a starting guess for the next optimization problem to improve convergence.

Application of this wider class of multipliers to the case of diagonal Δ follows similarly as before. This result will be communicated in future memos.

8. Application to Nonlinear Systems

In this section, some preliminary results on robustness of nonlinear systems (i.e., the state space description of $\mathcal{T}$ in Fig. 1 is nonlinear) are presented. To avoid too much technicality, we assume $\mathbf{X} = \mathbf{R}^n$. There are two possible approaches. The first approach converts the nonlinear state dynamics of $\mathcal{T}$ to a linear one with nonlinear perturbation and then apply results in sections 4–5. The second approach directly generalizes absolute and hyperstability to nonlinear systems by using the input/output characterization of passivity (Popov inequality) and an equivalent state space condition (nonlinear version of the Lur'e equations) [42]. Only the first approach will be discussed here, as the latter approach is still under development.

Suppose $x = 0$ is an equilibrium point (i.e., $f(t,0) = 0$) of the following system

$$\dot{x}(t) = f(t, x(t)) \quad . \tag{8.1}$$

We are interested in finding conditions for the equilibrium point to be globally asymptotically stable. Rewrite (8.1) as a linear system with a nonlinear perturbation:

$$\dot{x}(t) = -\alpha x(t) - (-f(t, x(t)) - \alpha x(t)) \quad , \tag{8.2}$$

where $\alpha > 0$ is an arbitrary scalar. Since $\dot{x} = -\alpha x$ is strictly positive real (with $A = -\alpha I$, $B = C = I$, $D = 0$, $P = I$, $\epsilon = 2\alpha$ and $Q = 0$ in the Lur'e equations), by Corollary 3

$$x^T f(t,x) \leq -\alpha \|x\|^2 \quad , \tag{8.3}$$

implies that (8.1) is exponentially stable. Contrast this condition with the condition in the Krasovskii Theorem [Theorem 6.4 in 62] (for time–invariant systems):

$$\nabla_x f(t,x) \leq -\delta I \quad . \tag{8.4}$$

We now show this is a special case of (8.3) . If (8.4) is satisfied, then there exist $\delta_1, \ldots, \delta_n$ such that

$$-\frac{\partial f_i(t,x)}{\partial x_i} \geq \delta_i > 0 \quad .$$

Integrate both sides from 0 to x_i. Then

$$\begin{cases} -f_i(t,x) \geq \delta_i x_i & \text{if } x_i > 0 \\ -f_i(t,x) \leq \delta_i x_i & \text{if } x_i < 0 \end{cases} \quad .$$

Hence, $-f_i(t,x)x_i \geq \delta_i x_i^2$, which implies (8.3) .

In the same spirit, the following more general result can be derived:

Proposition 6. Given (8.1) . Suppose that $f(t,x)$ is bounded with respect to x for each t and there exists a matrix $P > 0$ and a constant $\gamma > 0$ such that

$$-x^T P f(t,x) \geq \gamma \|x\|^2 \quad \text{for all } x \text{ and } t \quad . \tag{8.5}$$

Then (8.1) is exponentially stable.

Proof: Again rewrite (8.1) as (8.2). Let $z = P^{-1}x$ be the output of this system. Since $P > 0 \Rightarrow P^{-1} > 0$, it follows that $(-\alpha I, I, P^{-1}, 0)$ is strictly positive real. Define $f_1(t,z) \triangleq f(t, Pz) = f(t,x)$. By writing the perturbation term in terms of the output z, the following condition is sufficient for stability (from Corollary 3):

$$z^T(-f_1(t,z) - \alpha Pz) \geq 0 \quad . \tag{8.6}$$

We now show that (8.5) implies (8.6).

$$\begin{aligned}
&z^T(-f_1(t,z) - \alpha Pz) \\
=& - z^T f(t,x) - \alpha x^T P^{-1} x \\
=& - x^T P f(t,x) - \alpha x^T P^{-1} x \\
\geq& \gamma \|x\|^2 - \alpha \|P^{-1}\| \, \|x\|^2 \\
\geq& \lambda \|x\|^2 \quad , \lambda > 0 \quad .
\end{aligned}$$

The last inequality follows by setting α sufficiently small.

■

Note that condition (8.3) is a special case with $P = I$.

The same framework can be used to study robust stability of a perturbed nonlinear system. Suppose the actual system is of the form

$$\dot{x} = f_T(t,x) = f(t,x) + (f_T(t,x) - f(t,x)) \quad . \tag{8.7}$$

If the assumption in Proposition 6 is satisfied (implying that system (8.1) is exponentially stable), then a sufficient condition for stability is

$$-x^T P(f_T(t,x) - f(t,x)) \geq -\gamma \|x\|^2 \quad , \tag{8.8}$$

where γ is given by (8.5). By using the same technique as before, a condition that implies (8.8) is

$$-P(\nabla_x f_T - \nabla_x f) - (\nabla_x f_T - \nabla_x f)^T P + 2\gamma I \geq 0 \quad \text{for all } t \text{ and } x \quad . \tag{8.9}$$

This condition has appeared in [41].

Results in this section can be applied to a common situation: approximation of a nonlinear system by a linear system. As a special case, we will recover the result in Lemma 3.1 in [41]. Suppose the true plant is nonlinear but linear in the control and a full state feedback $u = -Gx$ has been applied. Then

$$\begin{aligned}
f(t,x) &= (A - BG)x \\
f_T(t,x) &= (A_T(t,x) - B_T(t,x)Gx) \quad .
\end{aligned}$$

Assume the linear system is closed loop stable. Then there exists $P > 0$ that satisfies

$$P(A - BG) + (A - BG)^T P = -Q < 0 \quad ,$$

which implies (8.5). Condition for the stability of the closed loop nonlinear system follows from (8.8):

$$\begin{aligned}
&x^T(P((Ax - A_T(t,x)) - (B - B_T(t,x))Gx) + ((Ax - A_T(t,x)) - (B - B_T(t,x))Gx)^T Px + x^T Qx \\
&\quad \geq 0 \quad \text{for all } t \text{ and } x \quad .
\end{aligned} \tag{8.10}$$

A sufficient condition, which is Lemma 3.1 in [41], follows from (8.9) :

$$P((A-\nabla_x A_T)-(B-\nabla_x(B_T Gx)))+((A-\nabla_x A_T)-(B-\nabla_x(B_T Gx)))^T P+Q \geq 0 \quad , \text{for all } t \text{ and } x \quad . \quad (8.11)$$

As stated earlier, the first approach is rooted in linear theory and the nonlinear dynamics is treated as a perturbation. It is possible to generalize the robustness analysis ideas presented in previous sections to nonlinear systems by using Popov inequality and nonlinear Lur'e equations to characterize dissipativeness. The ν–index can be defined based on the minimum feedforward required for the nonlinear system to satisfy the Popov inequality; and it can then be related to the nonlinear version of the absolute stability and hyperstability theorems. Past research in this area spearheaded by Moylan and Hill [42,63,64] has produced much of the required machinery. We will communicate our work in this area in the future.

9. Controller Synthesis by the Passivity Approach

Research into finding a stabilizing controller to attain the optimal ν–index is still at the preliminary stage. This section provides some ideas currently being explored. We will borrow heavily from [43] and the terminology in [43] will be used throughout this section. The idea is a straightforward one: use bilinear transform to convert the ν–index synthesis problem to an H_∞–norm synthesis problem, the solution of which is well known [44,65,43,66]. In certain cases, this approach yields nice analytical expressions for the achievable closed loop ν–index. One such case is when the open loop plant is additively perturbed and the ν–index of the closed loop transfer function around the perturbation is to be minimized. This problem can be transformed into the standard Nehari problem [65] which can be solved analytically. We then use this result to design a stabilizing finite dimensional compensator for an infinite dimensional open loop plant.

Assume a system configuration as given in Fig 8. Denote the open loop plant by G instead of P. The transfer function between w and z is given by the linear fractional transformation [65]

$$F_\ell(G,K) \triangleq [G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}] \quad . \quad (9.1)$$

Given (a,b), we say a transfer function $T \in \text{sector}[a,b]$ if T is stable and

$$\text{RF}\left[(I - \frac{1}{b}T)^*(T - aI)\right] \geq 0$$

where RF is as defined in (2.17) . The (a,b)–sector synthesis problem is defined as follows:

Given (a,b), find K such that $F_\ell(G,K)$ is in sector$[a,b]$.

As shown in [43],

$$\begin{aligned} &T \in \text{sector}[a,b] \\ \Leftrightarrow\, &Z \triangleq (T - aI)(I - b^{-1}T)^{-1} \in \text{sector}[0,\infty] \\ \Leftrightarrow\, &S \triangleq (I - Z)(I + Z)^{-1} \in \text{sector}[-1,1] \quad . \end{aligned}$$

From this result, it follows that the the (a,b)–sector synthesis problem for arbitrary (a,b) can be transformed to a $(-1,1)$–sector synthesis problem (also called the small gain synthesis problem) [43,Proposition 1].

Proposition 7. Given (a,b). K solves the (a,b)–sector synthesis problem if and only if K solves the $(-1,1)$–sector synthesis problem for the plant

$$\tilde{G} = \begin{bmatrix} M^{-1}(I - \Xi^{-1}(G_{11} - aI)) & -2(1 - ab^{-1})M^{-1}\Xi^{-1}G_{12} \\ G_{21}M^{-1}\Xi^{-1} & (b^{-1} - 1)G_{21}M^{-1}\Xi^{-1}G_{12} + G_{22} \end{bmatrix} , \tag{9.2}$$

where

$$\Xi = I - b^{-1}G_{11} \quad \text{and} \quad M = I + \Xi^{-1}(G_{11} - aI) \ .$$

■

To apply this result in our context, imagine an uncertainty $-\Delta$ that feeds from z to w. Assume a lowerbound, $\underline{\Delta}$, of Δ has been subtracted so that Δ is dissipative, and $\underline{\Delta}$ is also incorporated into the open loop plant (as a feedback). The problem of finding the upperbound of Δ can be posed as the following ν–index optimization problem:

Find K that stabilizes G and minimizes $\nu(F_\ell(G,K))$.

The above optimization problem corresponds to finding the smallest a such that $F_\ell(G,K) \in \text{sector}[-a,\infty]$. By Proposition 7, this problem is further equivalent to the small gain synthesis problem for the modfied plant:

$$\tilde{G} = \begin{bmatrix} -I + 2((1-a)I + G_{11})^{-1} & -2((1-a)I + G_{11})^{-1}G_{12} \\ G_{21}((1-a)I + G_{11})^{-1} & -G_{21}((1-a)I + G_{11})^{-1}G_{12} + G_{22} \end{bmatrix} . \tag{9.3}$$

The procedure of solving this problem is known [65,43 and references contained therein], we will only briefly outline the procedure below. By using the stable fractional representation of transfer functions, the complete set of stabilizing compensators can be parameterized by an RH_∞ (real proper stable transfer matrices) matrix, Q, called the Youla parameter. Then the problem of finding K so that $\|F_\ell(\tilde{G},K)\|_{H_\infty} \leq 1$ can be recast as a Hankel approximation problem of finding $X \in RH_\infty$ such that $\|R - X\|_{H_\infty} \leq 1$ for some $R \in RL_\infty$ [67] (real proper transfer matrices with no poles on $j\omega$–axis). The optimal value of $\|R - X\|_{H_\infty}$ equals to $\|\Gamma_R\|$, where Γ_R is the Hankel operator associated with R. $\|\Gamma_R\|$ can be easily computed [65], it is equal to the maximum eigenvalue of the product of the controllability and observability grammians of R. Therefore, the optimization problem is solvable if and only if $\|\Gamma_R\| \leq 1$. Formula for computing the optimal X can be found in, for example, [65,67,43].

For our original problem, we start from $a = 0$ and increase a until the problem becomes solvable (by checking if $\|\Gamma_R\| \leq 1$), at which time, find the optimal X for the Nehari problem and convert it back to the compensator.

In certain cases, a bound on a can be computed analytically. The rest of this section is devoted to one such important special case. Consider the following problem: Suppose the true plant is $P + \Delta P$ where P is known and $\Delta P \in RH_\infty$, find a finite dimensional K that stabilizes $P + \Delta P$. We first state a sufficient condition for a given K that stabilizes P to also stabilize $P + \Delta P$. This condition requires the product between the ν–index of the nominal closed loop transfer function around ΔP and the sum of ν–index and the H_∞–norm of ΔP to be sufficiently small. We then show the nominal ν–index only depends on the antistable part of P and a bound can be computed. If $P + \Delta P$ represents a high order (possibly infinite dimensional) open loop system and P is a lower order (finite dimensional) approximation to be used for low order controller design, the H_∞-norm of ΔP can be made small without affecting the nominal closed loop ν-index. Thus, the stability condition will be satisfied if either P is a high fidelity approximation of $P + \Delta P$ (small ΔP) or the antistable part of P can be stabilized and has a small ν-index with respect to the additive channel.

We first use hyperstability to derive a sufficient condition for a given compensator K to stabilize an additively perturbed system.

Proposition 8.

Suppose ΔP is exponentially stable and its upper and lower bounds are given as

$$\nu(\Delta P) \leq \gamma \quad , \gamma \geq 0 \qquad \|\Delta P\|_{H_\infty} \leq \delta \quad . \tag{9.4}$$

Let K be a compensator that stabilizes P. Define

$$\sigma \stackrel{\Delta}{=} \nu \left\{ K\left(I - (P - (\delta + 2\gamma) \cdot I)\,K\right)^{-1} \right\} \quad . \tag{9.5}$$

If

$$\sigma(\delta + \gamma) < \frac{1}{4} \quad , \tag{9.6}$$

then K also stabilizes $P + \Delta P$.

Proof:

Given a compensator K that stabilize P, decompose the closed loop system into two interconnected blocks as in Fig. 1, such that the forward system is $K\left(I + (P - (\delta + 2\gamma) \cdot I)K\right)^{-1}$ and the feedback system is $\Delta P + \delta + 2\gamma$. Recall that if an exponentially stable, LTI system is strictly positive real, then it satisfies the exponential Popov inequality. From Corollary 6 and Proposition 1, a condition for the interconnected system to be stable is

$$\sup_{\omega} z^*(\Delta P(j\omega) + \delta + 2\gamma)^*(\Delta P(j\omega) + \delta + 2\gamma - \frac{1}{\sigma})z < 0 \quad \text{for all } z \in \mathbf{C}^m \quad . \tag{9.7}$$

By using the upper and lower bounds of ΔP, it is clear that the inequality is implied by the condition (9.6), which completes the proof.

■

The nominal closed loop ν-index, σ, in (9.5) depends on ΔP (through γ and δ). If γ and δ are reduced through better approximation, σ may get worse. To derive an bound for σ independent of ΔP, first decompose the nominal open loop system to a stable part and an antistable part:

$$P - (\delta + 2\gamma) \cdot I = P_u + P_s - (\delta + 2\gamma) \cdot I \quad ,$$

where P_u is antistable and $P_s - (\delta + 2\gamma) \cdot I$ is stable. Without loss of generality, assume P_u is strictly proper, since any feedforward term can be absorbed in P_s. By subtracting the stable part from both $P - (\delta + 2\gamma) \cdot I$ and K, the nominal closed loop system can be written as (this transformation is motivated from [68])

$$K(I - (P - (\delta + 2\gamma) \cdot I)K)^{-1} = K_1(I - P_u K_1)^{-1} \quad , \tag{9.8}$$

where

$$K_1 = (I - K(P_s - (\delta + 2\gamma) \cdot I))^{-1} K \quad . \tag{9.9}$$

Since K_1 is an equivalent parameterization of K,

$$\inf_K \nu \left\{ K(I - (P - (\delta + 2\gamma) \cdot I)K)^{-1} \right\} = \inf_{K_1} \nu \left(K_1 (I - P_u K_1)^{-1} \right) \quad . \tag{9.10}$$

We now focus on the optimization in the right hand side of (9.10) and proceed to derive an upperbound that is independent of ΔP. We only outline the procedure here, the detail will be communicated in the future. The optimization problem in (9.10) is equivalent to finding the smallest a such that $(-a, \infty)$–sector synthesis problem associated with the plant

$$G = \begin{bmatrix} 0 & I \\ I & P_u \end{bmatrix} \quad , \tag{9.11}$$

is solvable. By Proposition 7, for a given a, the $(-a, \infty)$–sector problem is equivalent to the $(-1, 1)$–sector problem associated with the plant

$$\bar{G} = \begin{bmatrix} (2\zeta - 1)I & -2\zeta I \\ \zeta I & P_u - \zeta I \end{bmatrix} \qquad \zeta = \frac{1}{1+a} \quad . \tag{9.12}$$

For simplicity, we assume P_u does not have poles one the $j\omega$–axis. When the assumption is false, there is a simple modification which we will mention later in the section. This assumption is not overly restrictive, as we can always shift the $j\omega$–axis by a small amout to move these poles to the right half complex plane.

The complete set of stabilizing compensator for this plant is given by

$$K = (Y + MQ)(X + NQ)^{-1} \quad , \tag{9.13}$$

where M, N, X, Y and other relevant quantities are given by the following doubly coprime stable factorization of P_u, assuming $(A, B, C, 0)$ is a balanced (hence, minimal) realization of $P_u - \zeta I$ (for detail see [69]):

$$\begin{aligned}
P_u - \zeta I &= N M^{-1} = \tilde{M}^{-1} \tilde{N} \\
\begin{bmatrix} M & Y \\ N & X \end{bmatrix} &= \begin{bmatrix} I & 0 \\ -\zeta I & I \end{bmatrix} + \begin{bmatrix} -F \\ C + \zeta F \end{bmatrix} (sI - A + BF)^{-1} \begin{bmatrix} B & H \end{bmatrix} \\
\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} &= \begin{bmatrix} I & 0 \\ \zeta I & I \end{bmatrix} + \begin{bmatrix} F \\ -C \end{bmatrix} (sI - A + HC)^{-1} \begin{bmatrix} B + \zeta H & H \end{bmatrix} \\
A\Sigma + \Sigma A^T &= BB^T \\
A^T \Sigma + \Sigma A &= C^T C \\
\Sigma &= \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \quad \sigma_1 \geq \sigma_2 \ldots \geq \sigma_n > 0 \\
F = B^T \Sigma^{-1} & \qquad H = \Sigma^{-1} C^T \quad .
\end{aligned} \tag{9.14}$$

The two Lyapunov equations in (9.14) are solvable since we have assumed A does not have any purely imaginary eigenvalues [70]. With this choice of coprime factorization, M and $\tilde{M}$ are inners, i.e.,

$$M^* M = \tilde{M}^* \tilde{M} = I \quad .$$

After substituting the controller (9.13) into $F_\ell(\bar{G}, K)$, we obtain an equivalent problem (the model matching problem in [65]): Find $Q \in RH_\infty$ that satisfies

$$\|T_1 - T_2 Q T_3\|_{H_\infty} = \|F_\ell(\bar{G}, K)\|_{H_\infty} \leq 1 \quad , \tag{9.15}$$

where

$$\begin{aligned} T_1 &= (2\zeta - 1)I - 2\zeta^2 M\tilde{Y} \\ T_2 &= -2\zeta M \\ T_3 &= \zeta\tilde{M} \quad . \end{aligned} \tag{9.16}$$

The left hand side of (9.15) can be further manipulated into

$$\begin{aligned} \|T_1 - T_2 Q T_3\|_{H_\infty} &= 2\zeta^2 \|R - Q\|_{H_\infty} \\ R &= \left(\frac{2\zeta - 1}{2\zeta^2}\right) F_1 - F_2 \\ F_1 = M^* \tilde{M}^* \qquad & F_2 = \tilde{Y}\tilde{M}^* \quad . \end{aligned} \tag{9.17}$$

If M and $\tilde{M}$ are not inners due to $j\omega$–axis poles in P_u, (9.15) can still be manipulated into the standard Nehari problem by using the inner–outer factorization of M and co–inner–outer factorization of $\tilde{M}$ [65,§7]. Note that R is antistable, so this is the standard Nehari problem of finding the best stable approximation of an antistable L_∞ matrix. It is also important to note that F_1 and F_2 do not depend on ζ, since M, $\tilde{M}$ and $\tilde{Y}$ do not. Before stating the solution of this problem [71,67,65], some preliminaries are needed first. By using the expressions in (9.14) , we can compute a state space representation for F_1 and F_2:

$$\begin{aligned} F_1 &= I - [\,B^T \quad H^T\,]\left(sI - \begin{bmatrix} -(A+BF)^T & F^T H^T \\ 0 & -(A+HC)^T \end{bmatrix}\right)^{-1} \begin{bmatrix} F^T \\ C^T \end{bmatrix} \\ F_2 &= -F(sI - A)^{-1} H \quad . \end{aligned} \tag{9.18}$$

It is easy to see that the controllability grammian of F_1 is $\begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & \Sigma \end{bmatrix}$ and the observability grammian is $\begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma^{-1} \end{bmatrix}$, and, for F_2, both grammians are Σ^{-1}. Hence,

$$\begin{aligned} \|\Gamma_{F_1}\| = 1 \quad , \quad \|\Gamma_{F_2}\| = \frac{1}{\eta} \\ \eta \stackrel{\Delta}{=} \sigma_{\min}(P_u^*) = \text{ minimum Hankel singular value of } P_u^* \text{ [69]} \quad . \end{aligned} \tag{9.19}$$

The problem of finding Q to satisfy (9.15) has a solution if and only if

$$2\zeta^2 \|\Gamma_R\| \leq 1 \quad . \tag{9.20}$$

A sufficient condition is (since $\Gamma_R = \left(\frac{2\zeta-1}{2\zeta^2}\right)\Gamma_{F_1} + \Gamma_{F_2}$)

$$\begin{aligned} & 2\zeta^2 \left(\left(\frac{2\zeta - 1}{2\zeta^2}\right) \|\Gamma_{F_1}\| + \|\Gamma_{F_2}\|\right) \leq 1 \\ \Leftrightarrow \quad & a \geq \sigma^* \stackrel{\Delta}{=} \frac{1}{\frac{\eta}{2}\left(-1 + \sqrt{1 + \frac{4}{\eta}}\right)} - 1 \quad . \end{aligned} \tag{9.21}$$

In particular, if we choose $a = \sigma^*$, then the $(-\sigma^*, \infty)$ synthesis problem associated with G in (9.11) is solvable. Hence, a bound of the achievable ν–index in (9.10) is σ^*. The solution Q of (9.15) can be found in [67]. The corresponding controller K can then be computed from (9.9) and (9.13) .

In Proposition 8, σ in (9.6) can be replaced by σ^*, since there exists a compensator K such that in (9.5) $\sigma = \sigma^*$. If η is very large, then σ^* can be made close to zero and the stability condition in Proposition 8, (9.6) , is satisfied for very large ΔP ! To be more specific, if $\eta >> 4$, then $\sqrt{1+\frac{4}{\eta}} \approx 1+\frac{2}{\eta}$ and $\sigma^* \approx 0$. Hence, if η is large (much greater than 4), then only the unstable part of the high order system needs to be modeled for the compensator design (resulting in a lower order controller); otherwise, more stable portion needs to be incorporated into the nominal plant until ΔP becomes sufficiently small so that (9.6) is satisfied. This trade–off of achievable ν–index versus the order of controller is a unique feature of our approach in contrast to the small gain approach in [45].

There are two straightforward generalizations. So far, discussion is limited to square plants. The same analysis holds for a non–square plant if it is "squared up" with zeros. Another generalization is to modify $P+\Delta P$ by stable and stably invertible weightings, W_1 and W_2, so that the open loop plant is $W_1(P+\Delta P)W_2$ (the idea is used in [69]). This modification may affect the size of η.

10. Examples

Several examples are given in this section to illustrate several aspects of the results in this report. In the first example, we consider the linear quadratic regulator (LQR) problem. The well known $(\frac{1}{2},\infty)$ gain margin and $(-\frac{\pi}{3},\frac{\pi}{3})$ phase margin [22] in the control channel are demonstrated by showing the transfer function around perturbation in the control channel is positive real. Robustness issues related to parameter variations (ΔA and ΔB) are also discussed. The linear quadratic gaussian (LQG) controller is considered in the second example. The good robustness margin does not exist in general in this case. However, if certain transfer matrices are minimum phase, then loop transfer recovery (LTR) [72] method can be applied to approximately recover the margins in either the input or the output channel. The third example considers a one–dimensional heat equation. The first part deals with insulated boundaries. We design a one–mode stabilizing controller for this case. The second part deals with boundaries tied at a constant temperature. Two computational methods of the ν–index are compared: direct solution of a differential equation versus the finite dimensional approximation. The convergence result in Proposition 3 is demonstrated. The fourth and fifth examples illustrate directional robustness idea discussed in section 7. Two examples are taken from [73] which were originally used to illustrate the ability of the maximum singular value to detect vicinity of an instability region. Here, by deriving robustness margin in each quadrant, we show that robustness can be greatly improved if the nominal gains are changed. The last three examples address the use of multipliers. The first one is a simple harmonic oscillator with uncertain resonant frequency [16]. The second one has appeared in several papers on the Lyapunov–based robustness analysis [14,15]. The last example is from [74]. In each of these cases, we show that our approach gives superior results.

10.1 Robustness of Linear Quadratic Regulator

The linear quadratic optimal control problem is one of the most studied problem for infinite dimensional control systems. In this section, we will use absolute stability and hyperstability developed in previous sections to analyze robustness margins with respect to uncertainties in the control channel. This problem is well understood in finite dimensions, for example, [22] and [23] showed that finite dimensional linear quadratic optimal controllers possess $[-\frac{1}{2},\infty)$ gain margin and $[-\frac{\pi}{3},\frac{\pi}{3}]$ phase margin. We will generalize these results to evolution systems in Hilbert space.

Consider an evolution system in Hilbert space given by (1.1) . We will consider the full state feedback case, so $C=I$ and $D=0$. The input u is selected to minimize the following performance index:

$$J=\int_0^\infty \left(\langle Qx(t),x(t)\rangle + u(t)^T Ru(t)\right)\,dt \quad , \tag{10.1}$$

where $Q \geq 0$, $R \gg 0$ are bounded operators. From [75], [§4.4,47], the optimal controller is given by

$$u(t) = -R^{-1}\tilde{C}x(t) \quad , \quad \tilde{C} \triangleq B^{*}P \quad , \tag{10.2}$$

where P is the solution of the algebraic Riccati equation

$$(A^{*}P + PA + Q - PBR^{-1}B^{*}P)x = 0 \qquad \text{for all } x \in \mathcal{D}(A) \quad . \tag{10.3}$$

If (A, B) is exponentially stabilizable and $(A, Q^{\frac{1}{2}})$ is exponentially detectable, then (10.3) admits a unique positive solution P such that $A - BR^{-1}\tilde{C}$ generates an exponentially stable semigroup [Corollary 4.17 and Theorem 4.18, 47].

As in (2.6) , P can be written in the integral forms

$$Px = \int_0^{\infty} U_C^{*}(\tau)(Q + \tilde{C}^{*}R^{-1}\tilde{C})U_C(\tau)x\,d\tau \tag{10.4}$$

$$= \int_0^{\infty} U_{\eta C}^{*}(\tau)(Q + 2\eta\tilde{C}^{*}R^{-1}\tilde{C})U_{\eta C}(\tau)x\,d\tau \quad , \tag{10.5}$$

where $U_C(t)$ is the C_o–semigroup generated by $A - BR^{-1}\tilde{C}$ and $U_{\eta C}(t)$ is the C_o–semigroup generated by $A - (\frac{1}{2} + \eta)BR^{-1}\tilde{C}$. Following lemma shows that U_C and $U_{\eta C}$ are both exponentially stable C_o–semigroups.

Lemma 6. Given A, B as in (1.1) , $Q \geq 0$ and $R \gg 0$. If $(A, Q^{\frac{1}{2}})$ is detectable and there exists a self-adjoint, positive $P \in \mathcal{L}(\mathbf{X})$ and $G \in \mathcal{L}(\mathbf{X}, \mathbf{R}^m)$ such that

$$\langle P(A + BG)x, x\rangle + \langle x, P(A + BG)x\rangle + \langle Qx, x\rangle + \langle G^{*}SGx, x\rangle = 0 \quad , \tag{10.6}$$

for all $x \in \mathcal{D}(A)$. Then $(A + BG)$ generates an exponentially stable C_o–semigroup .

Proof: The proof is given in Appendix XI .

■

By setting $G = -R^{-1}\tilde{C}$ and $S = R$ in (10.6) , it follows from Lemma 6 that the Riccati equation (10.3) implies that U_C is exponentially stable. If $G = -(\frac{1}{2} + \eta)R^{-1}\tilde{C}$ and $S = 2\eta(\frac{1}{2} + \eta)^{-2}R$ in (10.6) , then (10.3) implies that $U_{\eta C}$ is exponentially stable for all $\eta > 0$.

The main result on the robustness margin can now be stated.

Theorem 5. Given the following linear time invariant system in a Hilbert space:

$$\dot{x} = Ax + B\mathcal{L}(-\tilde{C}x) \quad , x(0) = x_o \in \mathbf{X}, \tag{10.7}$$

where $\tilde{C}$ is defined by (10.2) . Assume that $(A, Q^{\frac{1}{2}})$ is exponentially detectable. Consider following classes of $\mathcal{L}$:

1. $\mathcal{L}(t, u) : \mathbf{R}_+ \times \mathbf{R}^m \to \mathbf{R}_+ \times \mathbf{R}^m$ is a bounded function for each t and

$$z^T(\mathcal{L} - \frac{1}{2}I)(R^{-1}z) \geq \mu \|z\|^2 \quad , \tag{10.8}$$

for some $\mu > 0$, all $t \in \mathbf{R}_+$ and all $z \in \mathbf{R}^m$.

2. $\mathcal{L}$ is an exponentially stable linear time invariant system and $(\mathcal{L} - \frac{1}{2}I)R^{-1}$ is strictly positive real.

3. $\mathcal{L} : L_{2_e}(\mathbf{R}^m) \to L_{2_e}(\mathbf{R}^m)$ is bounded in the sense that there exist constants γ_1, γ_2 such that

$$\|\mathcal{L}(z)\|_t \leq \gamma_1 + \gamma_2 \|z\|_t \quad , \tag{10.9}$$

and dissipative in the sense that there exist positive constants ξ_1 and ξ_2 such that

$$\left\langle (\mathcal{L} - \frac{1}{2}I)R^{-1}z, z \right\rangle_t \geq -\xi_1 + \xi_2 \|z\|_t^2 \quad . \tag{10.10}$$

If $\mathcal{L}$ belongs to the first two classes, then (10.7) is exponentially stable. If $\mathcal{L}$ belongs to the third class, then (10.7) is asymptotically stable.

Proof: Write (10.7) as

$$\dot{x} = (A - (\frac{1}{2} + \eta)BR^{-1}\tilde{C})x + B(\mathcal{L} - (\frac{1}{2} + \eta)I)(-R^{-1}\tilde{C}x) \quad . \tag{10.11}$$

This system can be represented in the interconnected form as in Fig. 1. The forward system is described by (1.1) with state space parameters $(A - (\frac{1}{2} + \eta)BR^{-1}\tilde{C}, B, \tilde{C}, 0)$. The feedback system is of the form $(\mathcal{L} - (\frac{1}{2} + \eta)I)(-R^{-1}\cdot)$. Since the forward system is exponentially stable by Lemma 6 and (10.3) implies that the Lur'e equations are satisfied with $\epsilon = 0$, the forward system is almost strictly positive real. By properties 3 and 4 of Proposition 9, ν–index of the forward system is non-positive. The stated result follows by applying Corollary 3for $\mathcal{L}$ in class 1, Corollary 5 and Remark 10 for $\mathcal{L}$ in class 2 and Corollary 6for $\mathcal{L}$ in class 3.

■

Remarks:

17. When $\mathcal{L}$ and R^{-1} are both diagonal, stability conditions in Theorem 5 can be stated in a more concise form. If $\mathcal{L}$ is linear and real, then the stability condition $\mathcal{L}_i \in (\frac{1}{2}, \infty)$ can be considered as the gain margin. If $\mathcal{L}_i = e^{j\phi}$, then the stability condition $\phi \in (-\frac{\pi}{3}, \frac{\pi}{3})$ can be considered as the phase margin. If each $\mathcal{L}_i$ is a linear time invariant system with Laplace transform $L_i(s)$, a condition for stability can be simply stated as

$$\nu(L_i) < \frac{1}{2} \quad .$$

■

Even though LQR offers impressive gain and phase margins, there is no guaranteed robustness margins against other types of perturbations, e.g., perturbation of A, B operators, directly. We will consider a simple example in [76] in which the stability margin with respect to a parameter in B can be made arbitrarily small. Let

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad R = r > 0 \qquad \Delta B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad . \tag{10.12}$$

The system is governed by

$$\dot{x} = Ax + Bu + \gamma \Delta B u \tag{10.13}$$

Nominally, $\gamma = 0$. Rewrite (10.13) as

$$\dot{x} = Ax + (B - \gamma_1 \Delta B)u + (\gamma + \gamma_1)\Delta B u \quad , \tag{10.14}$$

where $-\gamma_1$ is the design lower bound for γ. Let $u = -Gx$ be some controller such that $A - (B - \gamma_1 \Delta B)G$ is strictly stable. From the Corollary 3, G should be chosen to minimize the ν–index of the system T that has the internal parameters $(A - (B - \gamma_1 \Delta B)G, \Delta B, G, 0)$. Then $\gamma \in [-\gamma_1, \frac{1}{\nu(T)} - \gamma_1)$ preserves exponential stability. To ensure the nominal case, i.e., $\gamma = 0$, is included in the stability range, it is required $\frac{1}{\nu(T)} > \gamma_1$. Clearly, there is no reason why LQR design will always yield good margin since T is not guaranteed to be positive real. Indeed, as shown in the following table, the margin for the positive variation of γ is infinite but the margin for the negative variation becomes very poor as r becomes small (increasing performance).

r	$\nu(T)$	$\nu(-T)$
10^{-2}	0	1.43
10^{-3}	0	3.07
10^{-4}	0	6.12

Table. 1 The Lack of Robustness with respect to Input Matrix Uncertainty in LQR

This case demonstrates the advantage of a directional robustness measure, since the nominal value of γ can be chosen sufficiently large to provide arbitrary robustness about the nominal.

In this example, a different LQR design yields both good robustness margin and stability margin (in terms of distance from the $j\omega$–axis). Let α be the α–shift for the guaranteed closed loop stability margin [22]. Let the required underbound for γ be $-.9$ (it cannot be less than -1 since the α–shifted system becomes unstabilizable for some γ). The following table shows the robust stabilization design objective can be met:

γ_1	r	$\nu(T_\alpha)$
0.9	10^{-2}	0
0.9	10^{-3}	0

Table. 2 Robustness Margin with respect to Input Matrix Uncertainty

10.2 Robustness of Linear Quadratic Gaussian Controller

When only output is available for feedback, a state observer is typically used to reconstruct the state. By the separation principle, full state feedback control law used in conjunction with the estimated state stabilizes the open loop system. In this example, the LQR controller is used for the estimated state feedback and a Kalman filter type of design is used for the state estimator. Together, this combination is termed LQG controller. However, no stochastic connotation is intended here.

The Kalman filter type estimator is of the form

$$\begin{aligned} \dot{\hat{x}} &= (A + BG + KC)\hat{x} - Ky \\ u &= G\hat{x} \end{aligned} \tag{10.15}$$

where K, G are

$$\begin{aligned} G &= -R^{-1}B^T P = -R^{-1}\tilde{C} \\ K &= -HC^T N^{-1} \triangleq -\tilde{B}N^{-1} \end{aligned} \tag{10.16}$$

The matrices $P > 0$ solves ARE and $H > 0$ solves the dual algebraic Riccati equation (DARE):

$$AH + HA^T + M - HC^T N^{-1} CH = 0 \tag{10.17}$$

First consider a perturbation in the control channel. The true plant is now governed by

$$\dot{x} = Ax + B\mathcal{L}u$$

Close the loop with the LQG controller, the closed loop system is described by

$$\begin{aligned} \dot{x} &= Ax - BR^{-1}\bar{C}\hat{x} - B\mathcal{L}R^{-1}\bar{C}\hat{x} \\ \dot{\hat{x}} &= (A - BR^{-1}\bar{C} - \bar{B}N^{-1}C)\hat{x} + \bar{B}N^{-1}Cx \end{aligned} \tag{10.18}$$

In the state and estimator state error ($e = \hat{x} - x$) coordinate

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} &= \begin{bmatrix} A_G & -BR^{-1}\bar{C} \\ 0 & A_K \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} I \\ -I \end{bmatrix} B\tilde{u} \\ \tilde{u} &= -\mathcal{L}R^{-1}\tilde{y} \\ \tilde{y} &= \bar{C}[I \;\; I] \begin{bmatrix} x \\ e \end{bmatrix} \\ A_G = A - BR^{-1}\bar{C} & \qquad A_K = A - \bar{B}N^{-1}C \end{aligned} \tag{10.19}$$

By design, A_G and A_K are exponentially stable. In the transformed domain,

$$\begin{aligned} \tilde{y} &= \bar{C}\Phi_1\bar{B}N^{-1}C\Phi_2B\tilde{u} \\ &\triangleq T(s)\tilde{u} \end{aligned} \tag{10.20}$$

where $\Phi_1 = (sI - A_G)^{-1}$ and $\Phi_2 = (sI - A_K)^{-1}$. Since $(I - (1-\sigma)T(s)R^{-1})^{-1}T(s)$ is in general not positive real for $\sigma \in (\frac{1}{2}, \infty)$, the guaranteed stability margin no longer holds as in the LQR case. For a specific design, the robustness margins can be calculated as in section 5. When the plant is minimum phase, a method has been proposed in [9,72] to drive RF$[T](\omega)$ to zero over arbitrary range in ω. This technique has been termed as the loop transfer recovery / linear quadratic regulator (LTR/LQR) method. Suppose the estimator gain K is parameterized by a parameter q such that there exists some $W > 0$ such that $K(q) \to -BWq$ as $q \to \infty$. Then

$$\begin{aligned} T(j\omega) &= q\bar{C}\Phi_1BWC(sI - A + qBWC)^{-1}B \\ &= \bar{C}\Phi_1B(I + qWC(sI - A)^{-1}B)^{-1}qWC(sI - A)^{-1}B \end{aligned}$$

If q is sufficiently large over the bandwidth of $C(sI - A)^{-1}B$ (the open loop transfer function), then

$$T(j\omega) \approx \bar{C}\Phi_1B$$

which has been shown to be positive real in example 5. If this approximation holds true over the bandwidth of $\bar{C}\Phi_1B$, then the excursion of RF$[T](\omega)$ into the negative region is small, since $\bar{C}\Phi_1B$ is strictly proper. Given arbitrary ϵ_1 and ϵ_2, there exists q large enough (compared with both the open loop bandwidth and the closed loop LQR bandwidth) such that

$$\mathrm{RF}[T](\omega) \geq \begin{cases} \mathrm{RF}[\bar{C}\Phi_1B] - \epsilon_1 & \text{for } \omega \leq \omega_1 \\ -\epsilon_2 & \text{for } \omega > \omega_1 \end{cases}$$

Hence, $\nu(T) < \max(\epsilon_1, \epsilon_2)$. Similarly, $\nu((I - \frac{1}{2}TR^{-1})^{-1}T)$ can be made arbitrarily small by choosing q large. It is in this sense that the robustness margin of LQR is recovered. The estimator gain K with the above property can be achieved in the Kalman filter design by modifying the state noise covariance M to

$M + q^2BB^T$ and assume the plant is minimum phase. This fact follows directly from the cheap control and perfect regulation problems [78,79].

A similar technique exists for perturbations in the output channel, i.e., the true output is given by

$$y = \mathcal{L}Cx$$

It can be directly verified that in the closed loop

$$\tilde{y} = T(s)\tilde{u} \triangleq C\Phi_1 BR^{-1}\tilde{C}\Phi_2\tilde{B}\tilde{u}$$
$$\tilde{u} = -N^{-1}(\mathcal{L} - I)\tilde{y}$$

For minimum phase systems, the loop transfer recovery / Kalman–Bucy filter (LTR/KBF) technique was proposed in [80] to drive $T(s)$ close to positive real. This scheme uses the fact that $(I-(1-\sigma)C\Phi_2\tilde{B}N^{-1})^{-1}C\Phi_2\tilde{B}$ is positive real for $\sigma \in (\frac{1}{2}, \infty)$. To achieve the recovery in the same sense as the LTR/LQR case, the state penalty is modified to $Q + q^2C^TC$. Then, as $q \to \infty$, $G \to -qR^{\frac{1}{2}}C^T$ and $T(j\omega) \to C\Phi_2\tilde{B}$ pointwise in ω.

To demonstrate that $\nu(T)$ can be used as a measure of robustness margin with respect to perturbation in the control channel, even though the loop shape is far away from the LQR case, we use a simple example from [77]. Consider

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [1 \quad 0]$$
$$Q = q\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad R = 1 \quad M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad N = \sigma \tag{10.21}$$

Then

$$G = -[1 \quad 1]f \qquad f = 2 + \sqrt{4+q}$$
$$K = -\begin{bmatrix} 1 \\ 1 \end{bmatrix} d \qquad d = 2 + \sqrt{4+\sigma}$$

The actual system is assumed to be

$$\dot{x} = Ax + mBu$$

To ensure lower bound of m, rewrite the equation as

$$\dot{x} = Ax + m_1Bu + (m - m_1)Bu \quad , \quad m_1 > 0$$

Now design the LQG compensator for (A, m_1B, C). With the parameters as specified in [77], we have the following robustness margins for m (the actual margin and the prediction based on ν-index):

m_1	q	σ	$\nu(T)$ $(\nu(T)^{-1})$	$\nu(-T)$ $(\nu(-T)^{-1})$	guaranteed margin	actual margin
1	1	1	20 (0.05)	14.3 (0.07)	(0.93,1.05)	[0.92,1.05]
1	10	10	25 (0.04)	12.5 (0.08)	(0.92,1.04)	[0.92,1.04]
1	1000	1000	142 (0.007)	11.1 (0.09)	(0.91,1.007)	[0.91,1.007]

Table. 3 ν-Guaranteed Margin vs. Actual Margin, without LTR/LQR

We next apply the LTR/LQR technique to improve the stability margin. The results are summarized below:

m_1	q	σ	LTR parameter	$\nu(T)$ $(\nu(T)^{-1})$	$\nu(-T)$ $(\nu(-T)^{-1})$	guaranteed margin	actual margin
1	1	1	1	13.22 (0.07)	10.22 (0.09)	(0.91,1.07)	[0.91,1.07]
1	1	1	10	6.52 (0.15)	7.23 (0.14)	(0.86,1.15)	[0.86,1.15]
1	1	1	100	2.86 (0.35)	4.42 (0.23)	(0.77,1.35)	[0.77,1.35]
1	1	1	1000	1.36 (0.73)	3.09 (0.32)	(0.68,1.73)	[0.68,1.73]
0.5	1	1	10	1.32 (0.75)	5.22 (0.19)	(0.31,1.25)	[0.31,1.25]
0.5	1	1	10	0.21 (4.77)	4.14 (0.24)	(0.26,5.27)	[0.26,5.27]

Table. 4 ν–Guaranteed Margin vs. Actual Margin, with LTR/LQR

As indicated by these cases, the ν-guaranteed margin is almost identical to the actual margin.

10.3 One–Dimensional Heat Equation

10.3.1 Insulated Boundaries

In this example, we design a stabilizing compensator for a one–dimensional heat equation with insulated boundaries. Consider the following system

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + b(x)f(t) \\
&y(t) = < c, u > \\
&u_x(t,0) = u_x(t,1) = 0 \\
&\mathbf{X} = L_2[0,1] \quad , \quad A = \frac{\partial^2}{\partial x^2} \quad , \quad \mathcal{D}(A) = \{u \in H_2[0,1] : u_x(0) = u_x(1) = 0\} \quad .
\end{aligned} \tag{10.22}$$

Since A is of compact-normal resolvent, the solution can be put into the modal form

$$\begin{aligned}
u(t,x) &= \sum_{n=0}^{\infty} u_n(t)\phi_n(x) \qquad , \quad \phi_o(x) = 1 \quad , \quad \phi_n(x) = \sqrt{2}\cos n\pi x \\
\dot{u}_n &= -n^2\pi^2 u_n + b_n f \qquad\qquad b_n = < b, \phi_n > \\
y &= \sum_{n=0}^{\infty} c_n u_n \qquad\qquad c_n = < c, \phi_n > \quad .
\end{aligned}$$

Assume $c_o b_o \neq 0$ for stabilizability. This system has a single marginally stable mode, all the rest of the modes are in the open left half plane. We want to design a stabilizing controller based on an one-mode approximation. Write the infinite dimensional system in the perturbation form in the s–domain,

$$\begin{aligned}
y &= \frac{c_o b_o}{s} f + \sum_{n=1}^{\infty} \frac{c_n b_n}{s + n^2\pi^2} f \\
&= \left(\frac{|c_o b_o|}{\beta s} - \rho\right) u + \left(\rho + \frac{\operatorname{sgn}(c_o b_o)\Delta}{\beta}\right) u \quad ,
\end{aligned}$$

where $u = \operatorname{sgn}(c_o b_o)\beta f$, β is an arbitrary constant to be specified later, Δ is the exponentially stable unmodeled dynamics and ρ is an underbound for $\frac{\operatorname{sgn}(c_o b_o)\Delta}{\beta}$. Let the control law be a simple output feedback

$$u = -gy \quad .$$

Then the closed loop system can be written as

$$y = -\frac{g}{(1-\rho g)} \frac{s}{(s + \frac{|c_o b_o| g}{\beta(1-\rho g)})} (\frac{\text{sgn}\,(c_o b_o)\Delta}{\beta} + \rho) y \quad .$$

Clearly, if

$$0 < g < \frac{1}{\rho} \quad , \tag{10.23}$$

and $\text{sgn}\,(c_o b_o)\Delta + \rho\beta$ is strictly positive real, then the closed loop system is exponentially stable. Since Δ is exponentially stable, it is sufficient to have

$$\|\Delta\|_{H_\infty} < \rho\beta \quad . \tag{10.24}$$

Hence, if an upper bound of the H_∞ norm of the unmodeled dynamics is known, then for any $g > 0$, there exist β and ρ so that (10.23) and (10.24) are simultaneously satisfied. Finally, the one–mode stabilizing control law is given by

$$f = -\text{sgn}\,(c_o b_o) \frac{g}{\beta} y \quad . \tag{10.25}$$

Note that the magnitude of the feedback gain is bounded by $\frac{1}{\|\Delta\|_{H_\infty}}$.

10.3.2 Constant Temperature Boundaries

We now consider the a perturbed heat equation with both boundaries tied at a fixed temperature. The equation governing temperature evolution is given by

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} - b(x)\Delta(y) \\ y &= \langle c, u \rangle \\ \mathbf{X} &= L_2[0,1] \quad , \quad b,\, c \in \mathbf{X} \\ Au &\stackrel{\Delta}{=} \frac{\partial^2 u}{\partial x^2} \\ \mathcal{D}(A) &= \{u \in H_2([0,1]; \mathbf{R}) : u(0) = u(1) = 0\} \quad . \end{aligned} \tag{10.26}$$

Note that $H_2([0,1]; \mathbf{R})$ denotes the Sobolev space of functions whose first two generalized distributional derivatives are in $L_2([0,1]; \mathbf{R})$.

To find a class of Δ that preserves stability by using absolute stability and hyperstability, we regard the system described by (10.26) as a feedback interconnected system where the forward system has the state space parameters $(A, b, c, 0)$ and the feedback system is Δ. Assume Δ is locally Lipschitz. Then, as discussed in Section 4, a unique local mild solution exists.

We first consider the computation of the ν–index of the forward system by solving $\xi_\omega = (j\omega I - A)^{-1} b$. This is equivalent to finding ξ_ω that satisfies

$$j\omega\xi_\omega - \frac{\partial^2 \xi_\omega}{\partial x^2} = b(x) \quad , \quad \xi_\omega \in \mathcal{D}(A) \quad . \tag{10.27}$$

The solution is given by the variation of constant formula and the boundary conditions for elements in $\mathcal{D}(A)$

$$\xi_\omega(x) = \frac{\sinh\sqrt{j\omega}x}{\sqrt{j\omega}\sinh\sqrt{j\omega}} \int_0^1 \sinh\sqrt{j\omega}(1-\tau) b(\tau)\, d\tau - \frac{1}{\sqrt{j\omega}} \int_0^x \sinh(\sqrt{j\omega}(x-\tau)) b(\tau)\, d\tau \quad , \tag{10.28}$$

and the realness function is given by

$$\mathrm{RF}(c(sI-A)^{-1}b)(\omega) = \mathrm{Re}\,\frac{1}{\sqrt{j\omega}\sinh\sqrt{j\omega}}\int_0^1 c(x)\sinh\sqrt{j\omega}x\,dx\int_0^1 \sinh\sqrt{j\omega}(1-\tau)b(\tau)\,d\tau - \mathrm{Re}\,\frac{1}{\sqrt{j\omega}}\int_0^1 c(x)\int_0^x \sinh(\sqrt{j\omega}(x-\tau))b(\tau)\,d\tau\,dx \quad . \tag{10.29}$$

Alternatively, the realness function can also be computed by modal decomposition. The solution u can be expanded as

$$u(t,x) = \sum_{n=0}^{\infty} u_n(t)\phi_n(x)$$
$$\phi_n(x) = \sqrt{2}\sin\lambda_n x \quad , \quad \lambda_n = n\pi \quad . \tag{10.30}$$

By direct computation, we have

$$\xi_\omega = \sum_{n=0}^{\infty} \frac{\langle b,\phi_n\rangle\,\phi_n}{j\omega+\lambda_n^2} \quad . \tag{10.31}$$

Therefore,

$$\begin{aligned}\mathrm{RF}(c(sI-A)^{-1}b)(\omega) &= \mathrm{Re}\sum_{n=0}^{\infty}\frac{\langle c,\phi_n\rangle\langle b,\phi_n\rangle}{j\omega+\lambda_n^2} \\ &= \sum_{n=0}^{\infty}\frac{\lambda_n^2\langle c,\phi_n\rangle\langle b,\phi_n\rangle}{\omega^2+\lambda_n^4} \quad .\end{aligned} \tag{10.32}$$

The ν–index of the nominal forward system, $c(sI-A)^{-1}b$, can be computed by taking the infimum of the realness function computed by either methods described above. A class of Δ that maintains stability of (10.26) can be obtained by using results in Section 4. Note that if $b = c$, which means that the sensor and actuator are colocated, then from (10.32), the ν–index of the nominal system (with $\Delta = 0$) is non–positive. In this case, the nominal system is positive real and any bounded, non–negative Δ does not destabilize the system.

For a numerical example, let

$$c(x) = \begin{cases} 1 & x \in [.45, .55] \\ 0 & \text{otherwise} \end{cases}$$
$$b(x) = \begin{cases} 1 & x \in [.1, .15] \\ 0 & \text{otherwise} \end{cases} \quad .$$

The plot of the realness functions computed based on (10.29) and (10.32) (with 5–mode, 10–mode, 15–mode and 20–mode approximation) is shown in Fig. 9. Errors in the realness functions based on modal approximations are shown in Fig. 10. The ν–indices of these cases are shown in the table below:

Approximation	ν–index
5–mode	2.94×10^{-5}
10–mode	3.85×10^{-5}
15–mode	3.63×10^{-5}
20–mode	3.62×10^{-5}
exact	3.62×10^{-5}

Table. 5 Comparison of ν–indices

10.4 Diagonal Uncertainties: Example 1

Consider the following 2–input, 2–output plant

$$G(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s-100 & 10(s+1) \\ -10(s+1) & s-100 \end{bmatrix} \tag{10.33}$$

Suppose the loop is closed with negative feedback of the following form

$$\begin{bmatrix} 1+k_1 & 0 \\ 0 & 1+k_2 \end{bmatrix}$$

The objective is to characterize robustness margins with respect to k_1 and k_2. Rearrange the closed loop with transfer function T in the forward path and diag$\{k_1, k_2\}$ in the negative feedback path. Then

$$T = \frac{1}{(s+1)} \begin{bmatrix} 1 & 10 \\ -10 & 1 \end{bmatrix}$$

The ν–indices are four cases of positive or negative variations in k_1, k_2 are listed below:

k_1	k_2	ν–index	ν^{-1}
+	+	0	∞
-	-	1	1
-	+	10.05	0.1
+	-	10.05	0.1

Table. 6 Stability Margins for Example 4

The exact stability margin, the μ–measure based stability margin and the stability margins from Table 6 are shown in Fig.11. This example has been used in [73] to demonstrate the ability of the singular value type of robustness measure to detect closeness to instability. The H_∞ norm of T is 10.05 which corresponds to the worst case margin in Table 6. However, Table 6 provides much more information in terms of robustness margins in different directions of parameter variations. In this particular example, stability margins from Table 6 indicate that with different set of feedback gains (both k_1 and k_2 positive), arbitrary robustness with respect to k_1, k_2 can be attained.

10.5 Diagonal Uncertainties: Example 2

This example is also from [73]. Consider a two–input/two–output plant with transfer function

$$G(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} -47s+2 & 56s \\ -42s & 50s+2 \end{bmatrix} \tag{10.34}$$

Again close the loop with negative feedback of the following form

$$\begin{bmatrix} 1+k_1 & 0 \\ 0 & 1+k_2 \end{bmatrix}$$

where the nominal values of k_1, k_2 are zero. The objective is to characterize robustness margins with respect to k_1 and k_2. Rearrange the closed loop with transfer function T in the forward path and diag$\{k_1, k_2\}$ in the negative feedback path. The robust margin is listed in the table below.

k_1	k_2	ν–index	ν^{-1}
+	+	7.92	.13
-	-	8.42	.12
-	+	16.25	.06
+	-	0.50	2

Table. 7 Stability Margins for Example 5

The H_∞ norm of T is 16.3 which yields a margin for k_1, k_2 of ± 0.06. The exact stability margin, the μ–measure based stability margin and the stability margins from Table 7 are shown in Fig.12. Again, the ν-index based margins in Table 7 provides more information than the small gain based margin. It also points out the direction in which k_1 and k_2 should be moved (k_1 negative and k_2 positive) in order to enlarge the stability margin.

10.6 Multiplier Method: Example 1 (Damped Harmonic Oscillator)

Consider a simple damped harmonic oscillator with uncertain frequency:

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ -(1+\theta)\omega_o^2 & -\xi \end{bmatrix} x \\ &= \begin{bmatrix} 0 & 1 \\ -\omega_o^2 & -\xi \end{bmatrix} - \begin{bmatrix} 0 \\ \omega_o \end{bmatrix} [\omega_o \quad 0]\theta x \end{aligned} \tag{10.35}$$

where θ is nominally zero. The transfer function around θ is

$$T(s) = \frac{\omega_o^2}{s^2 + \xi s + \omega_o^2}$$

A straight forward calculation shows that

$$\begin{aligned}\nu(T) &= \frac{\omega_o^2}{(2\xi\omega_o + \xi^2)} \\ \nu(-T) &= 1\end{aligned}$$

Therefore, if

$$-\omega_o^2 < \theta\omega_o^2 < 2\xi\omega_o + \xi^2 \tag{10.36}$$

then the perturbed system remains exponentially stable. This is clearly very conservative, especially for lightly damped system.

If the multiplier $z_1(j\omega) = 1 + qj\omega$ is used, then so long as $q\xi \geq 1$, $\nu(Tz_1) \leq 0$. Hence, for all $\theta \in (-1, \infty)$, the system remains exponentially stable. By direct computation, we know this bound is non–conservative. The same bound is also obtained if the multiplier $z_2 = 1 + qj$ is used.

10.7 Multiplier Method: Example 2

The example in this subsection has been previously used by several authors to test their Lyapunov based robustness analysis methods [15,14]. Given

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \ .$$

We want to characterize a class of diagonal Δ such that $(A - BC + B\Delta C)$ is exponentially stable. The forward system in this case is

$$T = -C(sI - A + BC)^{-1}B \quad .$$

The stability margins, with and without multipliers, in each quadrant of the uncertain parameter space are shown in the table below:

k_1	k_2	$\nu(T)$	$\nu(z_1T)$	$\nu(z_2T)$
+	+	0.5806	0.5806	0.5806
−	+	0.3358	0.3358	0.3358
+	−	0.5739	0.5739	0.5739
−	−	0.0117	0	0

Table. 8 Robustness Margins in Example 7

The worst case quadrant gives a margin of $\frac{1}{0.5806} = 1.722$ which is better than all the previously published margins. The multiplier method reveals unlimited robustness margin in the $\{-,-\}$ quadrant.

10.8 Multiplier Method: Example 3

Next example has appeared in [74] and contains three uncertain elements. The system is given by

$$\begin{aligned} \dot{x} &= Ax + H_1\Delta_A H_2 x + Bu + F_1\Delta_B F_2 u \\ y &= Cx \\ u &= Ky \quad . \end{aligned}$$

The parameters are given as below:

$$A = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.010 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.707 & 1.4200 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.52 & 4.49 \\ 0 & 0 \end{bmatrix} \quad C = [0 \quad 1 \quad 0 \quad 0]$$

$$F_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} F_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} H_2 = [1 \quad 0]$$

The uncertainty Δ_A is assumed to be diagonal and Δ_B is a scalar constant.

Two controllers are stated in [74]: a nominal controller

$$K_o = \begin{bmatrix} -1.63522 \\ 1.582236 \end{bmatrix}$$

and a "robustified" controller

$$K^* = \begin{bmatrix} -0.99633989 \\ 1.801833665 \end{bmatrix} \quad .$$

For the nominal controller, the ν–indices in each quadrant, with and without multipliers, are listed below:

p_1	p_2	p_3	$\nu(T)$	$\nu(z_1T)$	$\nu(z_2T)$
+	+	+	1.41	0.7	0.81
−	+	+	1.45	0.78	1.46
+	−	+	1.13	0.67	0.23
−	−	+	1.14	0.28	0.33
+	+	−	1.37	0.63	0.59
−	+	−	1.44	0.74	0.73
+	−	−	1.17	0.87	0.24
−	−	−	1.18	0.33	0.23

Table. 9 Robustness Margins in Example 8, Nominal Controller Case

For the controller K^*, the ν–indices are listed below

p_1	p_2	p_3	$\nu(T)$	$\nu(z_1T)$	$\nu(z_2T)$
+	+	+	1.52	0.83	0.89
−	+	+	1.57	0.96	1.59
+	−	+	1.15	0.51	0.25
−	−	+	1.18	0.58	0.36
+	+	−	1.50	0.76	0.77
−	+	−	1.55	0.97	0.92
+	−	−	1.17	0.21	0.21
−	−	−	1.19	0.31	0.29

Table. 10 Robustness Margins in Example 8, Robustified Controller Case

The multiplier z_1 produces superior margins than z_2 in this case, though no general comparison can be made. The worst case margin in the K_o case is $\frac{1}{0.87} = 1.15$ and in the K^* case is $\frac{1}{0.97} = 1.03$. Both satisfy the specification, 0.0648, and are much better than the margins given in [74]. Ironically, our margin for the nominal case is better than the robustified case.

11. Conclusion

We have presented a new approach to robustness analysis and compensator synthesis for evolution systems by using a passivity approach. The abstract evolution equation setting is chosen so as to include applications to distributed parameter systems. Our results are based on the stability conditions involving the sector bounds of two interconnected, sector–bounded (in a general sense) systems, which are derived from the passivity theory (in the form of absolute stability and hyperstability). These conditions can be interpreted in the context of robustness analysis when one system is the nominal closed loop control system and the other the perturbation. When specialized to diagonally structured perturbations, the stability conditions are sharpened by using the multiplier technique. When, furthermore, each diagonal element is linear and constant, we introduce the concept of "directional robustness" which measures robustness in each quadrant of the perturbation parameter space. These ideas also have applications to nonlinear systems (i.e., nominal system is nonlinear), though the full generalization is not yet completed. In terms of controller synthesis, we use the fact that a sector synthesis problem can be converted to a small gain synthesis problem, for which the solution is known. This technique is applied to the finite dimensional compensator design for an infinite dimensional system. An intuitive and powerful result followed: If the unstable part of the open loop system can be stabilized such that certain transfer function is close to being passive, then only crude approximation of the stable part of the open loop system is needed for the design of a stabilizing compensator. If the smallest Hankel singular value of the conjugate of the unstable part is large, then the desired passivity property can be attained, meaning no information about the stable part is necessary for stabilizing compensator design. The full range of issues relating to controller synthesis based on the passivity approach remains to be fully explored, however, especially for the structured uncertainty case. As the many examples and applications in this report witness, the passivity approach presented here is a viable and useful tool for robustness analysis. Preliminary results also suggest its usefulness in the controller synthesis problem. It complements well existing small gain based techniques such as the H_∞-norm and μ-measure. Future agenda in this direction of research includes continuing investigation into the synthesis problem, especially for diagonally structured uncertainties, and generalization to unbounded input and output operators to allow consideration of boundary sensing and actuation.

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Appendix

Appendix I Proof of Lemma 1

Let $t \in [0,T]$, $T < \infty$. By using the semigroup property of $U(\cdot)$ and a change of variable, $V(x(t))$ can be written as:

$$\begin{aligned} V(x(t)) &= \int_0^\infty \left\langle R\left[U(t+\tau)x_o + \int_0^t U(t+\tau-s)Bu(s)\,ds\right], \right. \\ &\qquad \left. \left[U(t+\tau)x_o + \int_0^t U(t+\tau-s)Bu(s)\,ds\right]\right\rangle d\tau \\ &= \int_t^\infty \left\langle R\left[U(\tau)x_o + \int_0^t U(\tau-s)Bu(s)\,ds\right], \right. \\ &\qquad \left. \left[U(\tau)x_o + \int_0^t U(\tau-s)Bu(s)\,ds\right]\right\rangle d\tau \quad . \end{aligned} \tag{A.1.1}$$

We first show that the integrand of (A.1.1) belongs to L_1 for each $t \in \mathbf{R}$.

$$\begin{aligned} &\int_0^\infty \left|\left\langle R\left[U(\tau)x_o + \int_0^t U(\tau-s)Bu(s)\,ds\right], \right.\right. \\ &\qquad \left.\left. \left[U(\tau)x_o + \int_0^t U(\tau-s)Bu(s)\,ds\right]\right\rangle\right| d\tau \\ &\leq 2\|R\| \left[\frac{M^2\|x_o\|^2}{2\sigma} + \|B\|^2 M^2 \left(\frac{e^{2\sigma t}-1}{4\sigma^2}\right)\|u\|_t^2\right] \\ &\qquad \text{(by the Schwarz inequality)} \quad . \end{aligned}$$

By assumption, $u \in L_{2_e}$. Hence, the integral in (A.1.1) is absolutely continuous (therefore, differentiable) with respect to the its lower limit of integration. Denote the derivative by I_1, then

$$\begin{aligned} I_1 &= -\left\langle R\left(U(t)x_o + \int_0^t U(t-s)Bu(s)\,ds\right), \left(U(t)x_o + \int_0^t U(t-s)Bu(s)\,ds\right)\right\rangle \\ &= -\langle Rx(t), x(t)\rangle \quad . \end{aligned}$$

To show continuous differentiability with respect to t within the integrand in (A.1.1), we first note that for each $T \in \mathbf{R}_+$,

$$\begin{aligned} \int_0^T |U(\tau-s)Bu(s)|\,ds &\leq \|B\|\left[\int_0^T \|U(\tau-s)\|^2\,ds\right]^{\frac{1}{2}}\left[\int_0^T \|u(s)\|^2\,ds\right]^{\frac{1}{2}} \\ &< \infty \quad \text{(by the Schwarz inequality and the } L_{2_e} \text{ assumption on } u) \quad . \end{aligned}$$

Hence, $\int_0^t U(\tau-s)Bu(s)\,ds$ is absolutely continuous with respect to t for $t \in [0,T)$ and is therefore differentiable. Denote by I_2 the derivative of the integral in (A.1.1) with respect to t in the integrand, then by the chain rule

$$\begin{aligned} I_2 &= 2\int_t^\infty \left\langle RU(\tau-t)Bu(t), \left[U(\tau)x_o - \int_0^t U(\tau-s)Bu(s)\,ds\right]\right\rangle d\tau \\ &= 2\int_0^\infty \left\langle RU(\tau)Bu(t), \left[U(\tau+t)x_o + \int_0^t U(\tau+t-s)Bu(s)\,ds\right]\right\rangle d\tau \\ &\quad \text{(by a change of variable)} \\ &= 2\int_0^\infty \langle RU(\tau)Bu(t), U(\tau)x(t)\rangle\, d\tau \\ &= 2\langle PBu(t), x(t)\rangle \end{aligned}$$

Summarizing the above, we have: $V(x(t))$ is differentiable along the mild solution and its derivative $\dot{V}(t,x(t))$ is given by

$$\dot{V}(t,x(t)) = -\langle Rx(t), x(t)\rangle + 2\langle PBu(t), x(t)\rangle \quad .$$

■

Appendix II Proof of Fact 2

1. The factorization of PR systems is standard [81,53,25]. The second implication follows from the fact

$$\operatorname{Re} w^* U w = \|Vw\|^2$$

2. By definitions.

3. Follows from $\inf_\omega(ab(j\omega)) = |a| \inf_\omega(\operatorname{sign}(a)\, b(j\omega))$.

4. Follows from $\inf(a+b) \geq \inf a + \inf b$.

5. Follows from $[\mathrm{RF}(U+cI)](\omega) = c + [\mathrm{RF}(U)](\omega)$.

6. Follows from $|[\mathrm{RF}(\pm U)](\omega)| \leq \|U\|_{H_\infty}$ for all ω.

7. By definition

$$\nu_F(U) \geq -\lim_{\omega\to\infty} [\mathrm{RF}(T)](\omega) = 0 \quad .$$

8.

$$\begin{aligned}
\nu_F(K^* U K) &= -\inf_\omega \inf_{\|w\|=1} \operatorname{Re} w^* K^* U(j\omega) K w \\
&\leq -\inf_\omega \mu_{\min}(U(j\omega)) \sigma^2_{\min}(K) \\
&= \nu(U) \sigma^2_{\min}(K)
\end{aligned}$$

9. The inequality follows from

$$\begin{aligned}
\nu(U) &= \nu(V^* W V) \\
&\leq -\inf_\omega \mu_{\min}(W(j\omega)) \sigma^2_{\min}(V(j\omega)) \\
&\leq \nu(W) \inf_\omega \sigma^2_{\min}(V(j\omega))
\end{aligned}$$

10. The first equality follows from

$$\inf_{\|w\|=1} \operatorname{Re} w^* U(j\omega) K w = \inf_{\|Kw\|=1} \operatorname{Re} (Kw)^* K U(Kw)$$

For the second equality, first note that

$$\sup_{K \text{ Unitary}} \left[-\inf_\omega \inf_{\|w\|=1} \operatorname{Re} w^* U(j\omega) K w \right] = \sup_\omega \sup_{\substack{\|w\|=1 \\ \|z\|=1}} \operatorname{Re} w^* \left(-U(j\omega)\right) z$$

Clearly, the right hand side is bounded above by $\|U\|_{H_\infty}$. Given ϵ, let $W_1^* \Sigma W_2$ be the singular value decomposition of $-U(\omega)$ where $\|U(j\omega)\| = \|U\|_{H_\infty} - \epsilon$. If w and z are chosen so that $W_1 w$ and $W_1 z$

are the unit vectors with 1 as the first element. Then $\sup_{\omega} \sup_{\substack{\|w\|=1 \\ \|z\|=1}} \mathrm{Re}\, w^*(-U)w$ can be arbitrarily close to $\|U\|_{H_\infty}$. Hence, they are in fact equal. The last inequality follows similarly.

11. Clearly,

$$\inf_{\|w\|=1} \mathrm{Re}\, w^* U w = \inf_{\sum_i \|w_i\|^2=1} \sum_i w_i^* U_i w_i \geq \inf_{\sum_i \|w_i\|^2=1} \sum_i \mu_{\min}(U_i)\, \|w_i\|^2 = \min_i \mu_{\min}(U_i)$$

The lower bound can be attained by choosing $\|w_i\| = 1$ for i corresponding to the minimum $\mu_{\min}(U_i)$ and $\|w_i\| = 0$ for the rest. Hence, the inequality can be replaced by equality. After taking the negation of the infimum over all ω, the stated result follows.

12. From statements 2 and 4 above,

$$|\nu(U) - \nu(V)| \leq \nu(U - V) \leq \|U - V\|_{H_\infty}$$

■

Appendix III Proof of Proposition 3

We first state a simple lemma.

Lemma A.3.1.

Given complex matrices G_1, G_2, the following inequality holds:

$$\mu_{\min}(G_1) - \mu_{\min}(G_2) \leq \|G_1 - G_2\| \quad .$$

Proof: The inequality follows from direct manipulation:

$$\begin{aligned}
&\mu_{\min}(G_1) - \mu_{\min}(G_2) \\
&= \inf_{\substack{z \in \mathbf{C}^m \\ \|z\|=1}} \mathrm{Re}\ \langle G_1 z, z\rangle - \inf_{\substack{z \in \mathbf{C}^m \\ \|z\|=1}} \mathrm{Re}\ \langle G_2 z, z\rangle \\
&\leq \inf_{\substack{z \in \mathbf{C}^m \\ \|z\|=1}} \mathrm{Re}\ \langle G_1 z, z\rangle - \mathrm{Re}\ \langle G_2 v, v\rangle + \epsilon \\
&\quad \text{(Given any } \epsilon\text{, there exists such } v,\ \|v\| = 1.) \\
&\leq \mathrm{Re}\ \langle (G_1 - G_2)v, v\rangle + \epsilon \\
&\leq \|G_1 - G_2\| + \epsilon \quad .
\end{aligned}$$

Let $\epsilon \to 0$ to complete the proof.

■

Now we proceed with the proof of Proposition 3. The difference between the approximate transfer function, T_n, and the actual transfer function, T, can be overbounded in norm as below:

$$\begin{aligned}
&\|T_n(j\omega) - T(j\omega)\| \\
&\leq \|D_n - D\| + \frac{M}{\sigma}\|B\|\,\|C_n - C\| + \frac{M}{\sigma}\|C_n\|\,\|B_n - B\| \\
&\quad + \|C_n\|\,\|B_n\|\,\|((j\omega I - A)^{-1} - (j\omega I - A_n)^{-1})\| \\
&\quad \text{(by using (2.13))}
\end{aligned}$$

The first three terms converge to zero independent of ω. Let Ω be any compact set in $\mathbf{R}$. By Trotter–Kato Theorem [1,46], the last term converges to zero uniformly for $\omega \in \Omega$. Hence, $T_n(j\omega) \to T(j\omega)$ uniformly for $\omega \in \Omega$. By Lemma A.1, $\mathrm{RF}(T_n)(\omega) \to \mathrm{RF}(T)(\omega)$ uniformly for $\omega \in \Omega$.

∎

Appendix IV Proof of Theorem 2

<u>(2) $\Longrightarrow$ (1)</u>

Consider the optimization problem of finding $\hat{u} \in L_2((-\infty,\infty);\mathbf{R}^m)$ to minimize

$$J_f = \int_{-\infty}^{\infty} \{-\hat{x}^*(j\omega)F^T F\hat{x}(j\omega)^2 + 2\hat{u}^*(j\omega)\hat{y}(j\omega)\}\ d\omega$$

where the superscript * denotes complex conjugate transposition and $\hat{x}$, $\hat{y}$ and $\hat{u}$ are the Fourier transforms of x, y and u, respectively. By writing $\hat{x}$ in terms of the initial condition and the input, the optimization index can be expanded as

$$\begin{aligned}J_f = \int_{-\infty}^{\infty} \{ &- \left((j\omega I - A)^{-1}x_o + (j\omega I - A)^{-1}B\hat{u}(j\omega)\right)^* F^T F \left((j\omega I - A)^{-1}x_o + (j\omega I - A)^{-1}B\hat{u}(j\omega)\right)\\ &+ \hat{u}^*(j\omega)\left[(C(j\omega I - A)^{-1})B + D)^* + (C((j\omega I - A)^{-1})B + D)\right]\hat{u}(j\omega)\\ &+2\hat{u}^*(j\omega)C(j\omega I - A)^{-1}x_o\}\ d\omega\end{aligned}$$

Consider the problem as an L_2–optimization. Then

$$J_f = < R\hat{u}, \hat{u} > + < r, \hat{u} > + k$$

where the inner products are in the L_2 sense. A unique solution exists if R is a coercive $\mathcal{L}(L_2)$ (the space of bounded operators in L_2) operator. Now,

$$R = T^*(j\omega) + T(j\omega) - B^T(-j\omega I - A^T)^{-1}F^T F(j\omega I - A)^{-1}B$$

By condition (2), if

$$\eta > \|F(j\omega I - A)^{-1}B\|^2_{H_\infty} \tag{A.4.1}$$

then the operator R is coercive.

By the Plancherel Theorem [82], J_f can be transformed back to the time domain as

$$J = \int_0^{\infty} \left[-x(t)^T F^T Fx(t) + 2u^T(t)y(t)\right]\ dt$$

Since a unique solution of the optimal control problem exists, the necessary conditions from the Maximum Principle must be satisfied. The Hamiltonian is given by

$$H = -x^T F^T Fx + 2u^T(Cx + Du) + \lambda^T(Ax + Bu)$$

where λ is the costate or the Lagrange multiplier. The feed forward D in $u^T Du$ can be regarded as the symmetrized D. Since condition (2) implies $D > 0$, there exists $W > 0$ such that

$$D + D^T = W^T W$$

The optimal u is obtained by minimizing H:

$$u = -\frac{1}{2}W^{-1}W^{-T}(2Cx + B^T\lambda)$$

The costate equation is governed by

$$\dot{\lambda} = 2F^TFx - 2C^Tu - A^T\lambda$$

It can be shown [83] that λ depends linearly on x. Let

$$\lambda = -2Px$$

Then

$$\begin{aligned}(PA + A^TP + F^TF)x &= (C - B^TP)^Tu \\ &= -(C - B^TP)^TW^{-1}W^{-T}(C - B^TP)x\end{aligned}$$

Since the equality holds for all x, we have

$$\begin{aligned}PA + A^TP &= -F^TF - Q^TQ \\ C &= B^TP + WQ^T\end{aligned}$$

The first equation implies $P > 0$. By defining $L = F^TF$ where F is chosen positive definite and satisfies

$$\sigma^2_{\min}(F) < \frac{\eta}{\|(j\omega I - A)^{-1}B\|^2_{H_\infty}}$$

condition (1) is proved.

$(1) \Longrightarrow (2)$

(When $D > 0$)

Given the Lur'e equations, compute the Hermitian part of the transfer function as follows:

$$\begin{aligned}
&T(j\omega) + T^*(j\omega) \\
=&D + D^T + C(j\omega I - A)^{-1}B + B^T(-j\omega I - A^T)^{-1}C^T \\
=&W^TW + (B^TP - W^TQ)(j\omega I - A)^{-1}B + B^T(-j\omega I - A^T)^{-1}(PB - Q^TW) \\
=&W^TW + B^T(-j\omega I - A^T)^{-1}\left[(-j\omega I - A^T)P - P(j\omega I - A)\right](j\omega I - A)^{-1}B \\
&\quad - W^TQ(j\omega I - A)^{-1}B - B^T(-j\omega I - A^T)^{-1}Q^TW \\
=&W^TW + B^T(-j\omega I - A^T)^{-1}(Q^TQ + L)(j\omega I - A)^{-1}B - W^TQ(j\omega I - A)^{-1}B - B^T(-j\omega I - A^T)^{-1}Q^TW \\
=&(W^T + B^T(-j\omega I - A^T)^{-1}Q^T)(W + Q(j\omega I - A)^{-1}B) + B^T(-j\omega I - A^T)^{-1}L(j\omega I - A)^{-1}B \\
\geq&0
\end{aligned}$$

Assume condition (2) is false. Then there exist u_n, $\|u_n\| = 1$, and ω_n such that

$$0 \leq < (T(j\omega_n) + T^*(j\omega_n))u_n, u_n > \leq \frac{1}{n}$$

As $n \to \infty$, if $\omega_n \to \infty$, then

$$< (T(j\omega_n) + T^*(j\omega_n))\, u_n, u_n > \quad \to \quad < Du_n, u_n > \geq \eta > 0$$

which is a contradiction since the left hand side converges to zero. Hence, u_n and ω_n are both bounded sequences and therefore contain convergent subsequences u_{n_k} and ω_{n_k}. Let the limits be u_o and ω_o. Then

$$< (T(j\omega_o) + T^*(j\omega_o))\, u_o, u_o >= 0$$

This implies

$$\begin{aligned} Wu_o + Q(j\omega_o - A)^{-1}Bu_o &= 0 \\ L^{\frac{1}{2}}(j\omega_o I - A)^{-1}Bu_o &= 0 \end{aligned}$$

Since $L > 0$, the second equality implies

$$(j\omega_o I - A)^{-1}Bu_o = 0$$

Substituting back to the first equality yields

$$Wu_o = 0$$

The positive definiteness of W (by the assumption $D > 0$) implies contradiction. Hence, condition (2) is satisfied.

<u>(2) $\Longrightarrow$ (8)</u>

Since (2) $\Longrightarrow$ (1), the Lur'e equation holds. Let

$$V(x) = x^T P x$$

Then

$$\begin{aligned} \dot{V}(x(t)) &= x(t)^T PAx(t) + x(t)^T PBu(t) \\ &= -\frac{1}{2}x^T(t)Lx(t) - \frac{1}{2}\|Qx(t)\|^2 + u^T(t)Cx(t) + u^T(t)W^T Qx(t) \\ &= -\frac{1}{2}x^T(t)Lx(t) - \frac{1}{2}\|Qx(t)\|^2 - u^T(t)Du(t) + u^T(t)W^T Qx(t) + u^T(t)y(t) \\ &\leq -\frac{\epsilon}{2}\|x(t)\|^2 + u^T(t)y(t) - \frac{1}{2}\|Qx(t) - Wu(t)\|^2 \\ &\leq -\frac{\epsilon}{2}\|x(t)\|^2 + u^T(t)y(t) \end{aligned}$$

By integrating both sides, we have, for all $T \geq 0$

$$\int_0^T u^T(t)y(t)\,dt \geq -V(x_o) \tag{A.4.2}$$

Since (2.1) remains valid if D is replaced by $D - \epsilon$ for ϵ sufficiently small, (A.4.2) holds with y replaced by

$$y_1 = Cx + (D - \epsilon)u$$

Then (A.4.2) becomes

$$\int_0^T u^T(t)y(t)\,dt \geq \epsilon \int_0^T \|u(t)\|^2\,dt - V(x_o)$$

Identifying $-V(x_o)$ with $\xi(x_o)$ and ϵ with ρ, condition (8) follows.

<u>(8) $\Longrightarrow$ (2)</u>

Let $T \to \infty$ in (2.30), then

$$\int_0^\infty u^T(t)y(t)\,dt \geq \xi(x_o) + \rho \int_0^\infty \|u(t)\|^2\,dt$$

In particular, for $x_o = 0$,

$$\int_0^\infty u^T(t)y(t)\,dt \geq \rho \int_0^\infty \|u(t)\|^2\,dt$$

By the Plancherel Theorem,

$$\int_{-\infty}^\infty \hat{u}^*(j\omega)\hat{y}(j\omega)\,d\omega \geq \rho \int_{-\infty}^\infty \|\hat{u}(j\omega)\|^2\,d\omega$$

for all $\hat{u} \in L_2$. Suppose that for each $\eta > 0$, there exists $w \in \mathbf{C}$ and $\omega_o \in \mathbf{R}$ such that

$$w^*T(j\omega)w < \eta\,\|w\|^2$$

By the continuity of $w^*T(j\omega)w$ in ω, there exists an interval Ω around ω_o of length r such that

$$w^*T(j\omega)w < \eta\,\|w\|^2$$

for all $\omega \in \Omega$. Let

$$\hat{u}(j\omega) = \begin{cases} w & \text{if } \omega \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

Clearly, $\hat{u} \in L_2$. Then

$$\int_{-\infty}^\infty \hat{u}^*(j\omega)y(j\omega)\,d\omega = \int_{-\infty}^\infty \hat{u}^*(j\omega)T(j\omega)\hat{u}(j\omega)\,d\omega < r\eta\,\|w\|^2$$

and

$$\rho \int_{-\infty}^\infty \|\hat{u}(j\omega)\|^2\,d\omega = r\rho\,\|w\|^2$$

If $\eta < \rho$, this is a contradiction. Hence, there exists $\eta > 0$ such that (2.25) holds.

$\underline{(8) \Longrightarrow (11)}$

Condition (11) follows directly from condition (8).

$\underline{(11) \Longrightarrow (8)}$

The implication is obvious if $x_o = 0$. In the proof of (8) $\Rightarrow$ (2), x_o is taken to be zero. Therefore, for $x_o = 0$, (11) $\Rightarrow$ (8) $\Rightarrow$ (2). It has already been shown that (2) $\Rightarrow$ (8). Hence, (11) $\Rightarrow$ (2) $\Rightarrow$ (8).

$\underline{(1') \Longrightarrow (1)}$

By definition.

$\underline{(1) \Longrightarrow (1')}$

$\underline{(\text{if } D = 0)}$

If $D = 0$, then $W = 0$. Rewrite (2.1 a) as

$$A^T P + PA = -Q^T Q - L + 2\mu P - 2\mu P$$

For μ small enough,

$$Q^TQ + L - 2\mu P \geq 0$$

Hence, there exists Q_1 such that

$$A^TP + PA = -Q_1^TQ_1 - 2\mu P$$

Since (2.1 b) is independent of Q_1 when $D = 0$, (1′) is proved.

<u>(1′) $\Longleftrightarrow$ (6)</u>

By straightforward manipulation.

<u>(6) $\Longrightarrow$ (7)</u>

Same as in (1) $\Longrightarrow$ (2) except $L = 0$.

<u>(7) $\Longrightarrow$ (6)</u>

Standard positive realness lemma (see [53]).

<u>(4) $\Longrightarrow$ (7)</u>

By direct substitution

$$\begin{aligned}
&T(j\omega - \mu) + T^*(j\omega - \mu) \\
=&D + D^T + C(j\omega I - A - \mu I)^{-1}B + B^T(-j\omega I - A^T - \mu I)^{-1}C^T \\
=&T(j\omega) + T^*(j\omega) + \mu\left[C(j\omega I - A)^{-1}(j\omega I - A - \mu I)^{-1}B + B^T(-j\omega I - A^T - \mu I)^{-1}(-j\omega I - A^T)^{-1}C^T\right]
\end{aligned}$$

Therefore, for any $w \in \mathbf{C}^m$,

$$w^*T(j\omega - \mu)w \geq w^*T(j\omega)w - 2\mu\,\|C\|\,\|B\|\,\|(j\omega I - A)^{-1}\|\,\|(j\omega I - A - \mu I)^{-1}\|\,\|w\|^2$$

Since

$$\|(j\omega I - A)x\| \geq |(\,|\omega| - \|A\|)\,\|x\||$$

It follows [56]

$$\|(j\omega I - A)^{-1}\| \leq \frac{1}{|\,|\omega| - \|A\||}$$

Then

$$w^*T(j\omega - \mu)w \geq w^*T(j\omega)w - \frac{2\mu\,\|C\|\,\|B\|\,\|w\|^2}{|\,|\omega| - \|A\||\,|\,|\omega| - \|A + \mu I\||}$$

By (2.27 a), for all $\omega \in \Omega$, Ω is compact in $\mathbf{R}$, there exists $k > 0$, k dependent on Ω, such that

$$w^*T(j\omega)w \geq k\,\|w\|^2 \tag{A.4.3}$$

By (2.27 b), for ω sufficiently large, there exists $g > 0$ such that

$$w^*T(j\omega)w \geq \frac{g}{\omega^2}\,\|w\|^2 \tag{A.4.4}$$

Hence, there exists $\omega_1 \in \mathbf{R}$ large enough so that (A.4.3) and (A.4.4) hold with some g and k dependent on ω_1. Then, for $|\omega| \leq \omega_1$,

$$\begin{aligned}
w^*T(j\omega - \mu)w &\geq k\,\|w\|^2 - \frac{2\mu\,\|C\|\,\|B\|\,\|w\|^2}{|\,|\omega| - \|A\||\;|\,|\omega| - \|A + \mu I\||} \\
&\geq k\,\|w\|^2 - \mu\left\{\sup_{\|\omega\| \leq \omega_1} \frac{2\,\|C\|\,\|B\|\,\|w\|^2}{|\,|\omega| - \|A\||\,|\,|\omega| - \|A + \mu I\||}\right\}
\end{aligned} \tag{A.4.5}$$

and for $\omega > \omega_1$

$$w^* T(j\omega - \mu) w \geq \frac{g}{\omega^2} \|w\|^2 - \frac{2\mu \|C\| \|B\| \|w\|^2}{\left| |\omega| - \|A\| \right| \left| |\omega| - \|A + \mu I\| \right|}$$
$$\geq \frac{\|w\|^2}{\omega^2} \left(g - \mu \left\{ \sup_{|\omega| > \omega_1} \frac{2 \|C\| \|B\| \|w\|^2}{\left| |\omega| - \|A\| \right| \left| |\omega| - \|A + \mu I\| \right|} \right\} \right) \tag{A.4.6}$$

The terms in curly brackets in (A.4.5) and (A.4.6) are finite. Hence, there exists μ small enough such that (A.4.5) and (A.4.6) are both non-negative, proving condition (7).

<u>(7) $\Longrightarrow$ (4)</u>

From (7) $\Longrightarrow$ (6), the minimal realization (A, B, C, D) associated with $T(j\omega)$ satisfies the Lur'e equation with $L = 2\mu P$. Following the same derivation as in (1) $\Longrightarrow$ (2), for all $w \in \mathbf{C}^m$, we have

$$\begin{aligned}
&w^*(T(j\omega) + T^*(j\omega))w \\
=&w^*(W^T + B^T(-j\omega I - A^T)^{-1}Q^T)(W + Q(j\omega I - A)^{-1}B)w \\
&\quad + 2\mu w^* B^T(-j\omega I - A^T)^{-1} P (j\omega I - A)^{-1} B w \\
\geq& 2\mu w^* B^T(-j\omega I - A^T)^{-1} P (j\omega I - A)^{-1} B w \\
\geq& \frac{2\mu \mu_{\min}(P) \sigma_{\min}(B)}{\left| |w| - \|A\| \right|^2} \|w\|^2
\end{aligned}$$

Since P is positive definite and, by assumption, $\sigma_{\min}(B) > 0$, $T(j\omega)$ is positive for all $\omega \in \mathbf{R}$.

It remains to show (2.27 b). Multiply both sides of the inequality above by ω^2, then

$$\omega^2 w^*(T(j\omega) + T^*(j\omega))w \geq \frac{\omega^2 2\mu \mu_{\min}(P) \sigma_{\min}(B)}{\left| |w| - \|A\| \right|^2} \|w\|^2$$

As $\omega^2 \to \infty$, the lower bound converges to $2\mu \mu_{\min}(P) \sigma_{\min}(B)$ which is positive.

<u>(7) $\Longrightarrow$ (5)</u>

If (2.28) is satisfied, $T(j\omega - \mu)$ corresponds to the driving point impedance of a multiport passive network [53]. Hence, $T(j\omega)$ corresponds to the impedance of the same network with all C replaced by C in parallel with a resistor of conductance μC and L replaced by L in series with a resistor of resistance μL. Since all L, C elements are now lossy, or dissipative, $T(j\omega)$ is the driving point impedance of a dissipative network.

<u>(5) $\Longrightarrow$ (7)</u>

Reversing the above argument, if $T(j\omega)$ is the driving point impedance of a dissipative network, all L and C elements are lossy. Hence, by removing sufficiently small series resistance in L and parallel conductance in C, the network would remain passive. Hence, again by [53], condition (7) is satisfied.

<u>(6) $\Longrightarrow$ (9)</u>

Let

$$V(t, x) = e^{\gamma t} x^T P x$$

Then

$$\begin{aligned}
\dot{V}(t, x(t)) &= \gamma e^{\gamma t} x^T(t) P x(t) + \frac{1}{2} e^{\gamma t} x^T(t)(PA + A^T P)x(t) + e^{\gamma t} x^T(t) P B u(t) \\
&\leq \gamma V(t, x(t)) - \frac{\epsilon}{2} \frac{V(t, x(t))}{\|P\|} - e^{\gamma t} \|Qx(t) - Wu(t)\|^2 + e^{\gamma t} u^T(t) y(t) \\
&\leq -\left(\frac{\epsilon}{2\|P\|} - \gamma \right) V(t, x(t)) + e^{\gamma t} u^T(t) y(t)
\end{aligned}$$

Choose $0 < \gamma < \frac{\epsilon}{2\|P\|}$. Then by comparison principle, for all $T \geq 0$,

$$\int_0^T e^{\gamma t} u^T(t) y(t)\, dt \geq -x_o^T P x_o$$

<u>(9) $\Longrightarrow$ (6)</u>

Define

$$\begin{aligned} u_1(t) &= e^{\frac{\gamma}{2}t} u(t) \\ y_1(t) &= e^{\frac{\gamma}{2}t} y(t) \\ x_1(t) &= e^{\frac{\gamma}{2}t} x(t) \end{aligned} \tag{A.4.8}$$

where $\gamma > 0$ is as given in (2.30) . Then

$$\begin{aligned} \dot{x}_1 &= (A + \frac{\gamma}{2} I) x_1 + B u_1 \\ y_1 &= C x_1 + D u_1 \end{aligned}$$

The corresponding transfer function is

$$\begin{aligned} T_1(j\omega) &= D + C(j\omega I - A - \frac{\gamma}{2} I)^{-1} B \\ &= T(j\omega - \frac{\gamma}{2}) \end{aligned}$$

By setting $T = \infty$ and $x_o = 0$ in (2.30) ,

$$\int_0^\infty u_1^T(t) y_1(t)\, dt \geq 0$$

By the Plancherel Theorem,

$$\int_{-\infty}^{\infty} \hat{u}_1^*(j\omega)(T_1(j\omega) + T_1^*(j\omega))\hat{u}_1(j\omega)\, d\omega \geq 0$$

Since this holds true for all $\hat{u}_1(j\omega) \in L_2$,

$$T_1(j\omega) + T_1^*(j\omega) \geq 0$$

Equivalently,

$$T(j\omega - \frac{\gamma}{2}) + T^*(j\omega - \frac{\gamma}{2}) \geq 0$$

proving (7).

<u>(9) $\Longrightarrow$ (10)</u>

Use the transformation in (A.4.8) , then condition (10) follows directly from condition (9) with $\alpha = \frac{\gamma}{2}$.

<u>(10) $\Longrightarrow$ (9)</u>

If $x_o = 0$, (10) $\Rightarrow$ (9) is obvious. Since in the proof of (9) $\Rightarrow$ (6), only $x_o = 0$ case is considered, it follows, for the $x_o = 0$ case, (10) $\Rightarrow$ (9) $\Rightarrow$ (6). It has already been shown that (6) $\Rightarrow$ (9). Hence, (10) $\Rightarrow$ (6) $\Rightarrow$ (9).

<u>(2) $\Longrightarrow$ (4) $\Longrightarrow$ (3)</u>

The implications are obvious.

■

Appendix V Proof of Corollary 2

In (A.4.1) in the proof of Theorem 2(Appendix 4), after substituting $\epsilon I + \gamma C^T C$ into $F^T F$, we have the following sufficient condition for condition 1:

$$\epsilon \|(j\omega I - A)^{-1} B\|^2 + \gamma \|T_1(j\omega)\|^2 < \eta \quad \text{for all } \omega \in \mathbf{R}. \tag{A.5.1}$$

It is straightforward to show that (A.5.1) follows from (2.35) - (2.36) .

■

Appendix VI Proof of Proposition 4

The first five conditions are standard. The proof of their equivalence can be found in, for example, [53], [50]. The equivalence of condition (6) to the rest will be shown here.

(2) $\Longrightarrow$ (6)

By condition (2) , $D \geq 0$. The transfer function of $(A, B, C, D + \rho I)$ is $T(j\omega) + \rho I$. Since

$$T(j\omega) + T^*(j\omega) + 2\rho I \geq 2\rho I$$

and $D + \rho I > 0$, the Lur'e equations are satisfied by Lemma 1. Hence, condition (6) is true.

(6) $\Longrightarrow$ (2)

From condition (6) and Lemma 1, there exists $\eta_n > 0$, monotonically decreasing, such that for all $w \in \mathbf{C}^m$,

$$w^*(T(j\omega) + T^*(j\omega) + \frac{1}{n})w \geq \eta_n \|w\|^2$$

As $n \to \infty$, $\eta_n \to \eta \geq 0$ and $\frac{1}{n} \to 0$. Hence,

$$T(j\omega) + T^*(j\omega) \geq 0$$

proving condition (2).

■

Appendix VII Proof of Lemma 2

Define

$$z(t) = \int_0^t U(t-s)Bu(s)ds \quad . \tag{A.7.1}$$

1. Let $x_o = 0$. By assumption, $z \in L_2(\mathbf{R}_+; \mathbf{X})$. Suppose $U(t)$ is not exponentially stable. By Datko's theorem, $U(\cdot)v \notin L_2(\mathbf{R}_+; \mathbf{X})$ for some $v \in \mathbf{X}$. Consider (1.2) with $x_o = v$; we have an L_2-function as the sum of a non-L_2-function, $U(t)v$, and an L_2-function, $z(t)$, which is a contradiction. Hence, $U(t)$ is exponentially stable.

2. Since $U(t)$ is exponentially stable from part 1, it suffices to show the convergence of $z(t)$ to zero. Equation (A.7.1) can be written as

$$\begin{aligned} z(t) &= \int_0^{\frac{1}{2}t} U(t-s)Bu(s)ds + \int_{\frac{1}{2}t}^t U(t-s)Bu(s)ds \\ &= \int_{\frac{1}{2}t}^t U(s)Bu(t-s)ds + \int_{\frac{1}{2}t}^t U(t-s)Bu(s)ds \quad . \end{aligned}$$

Overbound $z(t)$ by using the Schwarz inequality and the exponential bound of $U(t)$ given by (1.3), we have

$$\begin{aligned}
\|z(t)\| &\le \left[\int_{\frac{1}{2}t}^{t} M^2 e^{-2\sigma s}\|B\|^2\,ds\right]^{\frac{1}{2}}\left[\int_0^{\frac{1}{2}t}\|u(s)\|^2\,ds\right]^{\frac{1}{2}} \\
&\quad + \left[\int_{\frac{1}{2}t}^{t} M^2 e^{-2\sigma(t-s)}\|B\|^2\,ds\right]^{\frac{1}{2}}\left[\int_{\frac{1}{2}t}^{t}\|u(s)\|^2\,ds\right]^{\frac{1}{2}} \\
&\le \left[\int_{\frac{1}{2}t}^{\infty} M^2 e^{-2\sigma s}\|B\|^2\,ds\right]^{\frac{1}{2}}\left[\int_0^{\infty}\|u(s)\|^2\,ds\right]^{\frac{1}{2}} \\
&\quad + \frac{M\,\|B\|}{\sqrt{2\sigma}}\left[\int_{\frac{1}{2}t}^{\infty}\|u(s)\|^2\,ds\right]^{\frac{1}{2}} \quad .
\end{aligned}$$

Since $e^{-2\sigma t} \in L_2(\mathbf{R}_+;\mathbf{R})$ and $u \in L_2(\mathbf{R}_+;\mathbf{R}^m)$, the right hand side tends to zero as $t \to \infty$.

■

Appendix VIII Proof of Lemma 3

1. By integrating both sides of (3.1), we have

$$\int_0^{\infty}\|x(t)\|^2\,dt \le \frac{1}{\epsilon}V(x(0)) < \infty \quad .$$

Then it follows from Lemma 2 that $x(t) \to 0$ as $t \to \infty$.

2. Let

$$V_1(t,x) = e^{2\sigma t}V(x) \qquad , \qquad \sigma < \epsilon\,\|P\|^{-1} \quad .$$

The derivative along the solution trajectory is

$$\begin{aligned}
\dot{V}_1(t,x(t)) &= 2\sigma e^{2\sigma t}V(x(t)) + e^{2\sigma t}\dot{V}(x(t)) \\
&\le e^{2\sigma t}(2\sigma\,\|P\| - \epsilon)\,\|x(t)\|^2 \\
&= -\lambda e^{2\sigma t}\,\|x(t)\|^2 \qquad , \qquad \lambda > 0 \quad .
\end{aligned}$$

By integrating both sides, we get

$$\begin{aligned}
\int_0^t e^{2\sigma\tau}\,\|x(\tau)\|^2\,d\tau &\le \frac{1}{\lambda}V(x(0)) - \frac{1}{\lambda}e^{2\sigma t}V(x(t)) \\
&\le \frac{1}{\lambda}V(x(0)) < \infty \quad .
\end{aligned}$$

This shows that $e^{\sigma t}x(t) \in L_2$.

3. Since $e^{\sigma t}x(t) \in L_2$ by part 2 and $e^{\sigma t}u(t) \in L_2$ by assumption, we can apply Lemma 2, with $U(t)$, $x(t)$, $u(t)$, replaced by $e^{\sigma t}U(t)$, $e^{\sigma t}x(t)$, $e^{\sigma t}u(t)$, respectively, to show that there exists $M(x_o) < \infty$ such that $e^{\sigma t}\,\|x(t)\| \le M(x_o)$. This is equivalent to $\|x(t)\| \le M(x_o)e^{-\sigma t}$ for all $t \ge 0$.

■

Appendix IX   Proof of Lemma 4

If Δ is non-negative, then

$$\begin{aligned} 0 &\leq -u^T y \\ &= -u^T(Cx + Du) \\ &\leq \|u\|\,\|Cx\| - \mu_{\min}(D)\,\|u\|^2 \\ &= -\mu_{\min}(D)\left(\|u\| - \frac{\|Cx\|}{2\mu_{\min}(D)}\right)^2 + \frac{\|Cx\|^2}{4\mu_{\min}(D)} \quad . \end{aligned}$$

It then follows that there exists $\eta > 0$ such that

$$\|u(t)\| \leq \eta\,\|x(t)\| \tag{A.9.1}$$

for all $t \geq 0$. The mild solution can now be bounded as follows:

$$\begin{aligned} e^{\sigma t}\,\|x(t)\| &\leq M\,\|x_o\| + M\,\|B\| \int_0^t e^{\sigma\tau}\,\|u(\tau)\|\,d\tau \\ &\leq M\,\|x_o\| + \eta M\,\|B\| \int_0^t e^{\sigma\tau}\,\|x(\tau)\|\,d\tau \quad . \end{aligned}$$

Apply the Gronwell inequality, we get

$$\|x(t)\| \leq M\,\|x_o\| e^{-(\sigma - \eta M\,\|B\|)t} \quad .$$

Hence, x does not finitely escape and $x \in L_{2_e}(\mathbf{X})$. $u \in L_{2_e}(\mathbf{R}^m)$ follows from (A.9.1) .

∎

Appendix X   Proof of Lemma 5

By assumption, Δ satisfies the Popov inequality. Therefore, there exists $\xi > 0$ such that

$$\xi \geq \langle u, Cx + Du\rangle_t \geq \mu_{\min}(D)\left(\|u\|_t - \frac{\|Cx\|_t}{2\mu_{\min}(D)}\right)^2 - \frac{\|Cx\|_t^2}{4\mu_{\min}(D)} \quad .$$

It then follows easily that there exist positive constants η_1 and η_2 such that

$$\|u\|_t \leq \eta_1 + \eta_2\,\|x\|_t \quad . \tag{A.10.1}$$

Let $t \in [0, T)$, $T < \infty$. The mild solution of T can be bounded by

$$\|x(t)\| \leq Me^{-\sigma t}\,\|x_o\| + M\,\|B\| \int_0^t e^{-\sigma(t-\tau)}\,\|u(\tau)\|\,d\tau \quad . \tag{A.10.2}$$

Squaring both sides, we have

$$\begin{aligned} \|x(t)\|^2 &\leq 2M^2\,\|x_o\|^2 + 2M^2\,\|B\|^2 \left[\int_0^t \|u(\tau)\|\,d\tau\right]^2 \\ &\leq 2M^2\,\|x_o\|^2 + 2M^2\,\|B\|^2 T \int_0^t \|u(\tau)\|^2\,d\tau \\ &\quad \text{(by the Schwarz inequality)} \\ &\leq 2M^2\,\|x_o\|^2 + 2M^2\,\|B\|^2 T\left(2\eta_1^2 + 2\eta_2^2 \int_0^t \|x(\tau)\|^2\,d\tau\right) \quad . \end{aligned}$$

By applying the Gronwell inequality, it follows that x does not finitely escape.

It also follows from (A.10.2) that there exist positive constants a and b such that

$$\begin{aligned}\|x\|_t^2 &\leq a + b\int_0^t \left[\int_0^s \|u(\tau)\|\,d\tau\right]^2 ds \\ &\leq a + b\int_0^t s\cdot \|u\|_s^2\,ds \\ &\leq a + 2b\eta_1^2 T + 2b\eta_2^2 T\int_0^t \|x\|_s^2\,ds \quad .\end{aligned}$$

From the Gronwell inequality, we have $x \in L_{2_e}$. $u \in L_{2_e}$ follows from (A.10.1) .

If Δ satisfies the exponential Popov inequality, identical steps as in the first part of the proof can be followed with $x(t)$ and $u(t)$ replaced by $e^{\sigma t}x(t)$ and $e^{\sigma t}u(t)$, respectively, to show that $e^{\sigma t}x(t) \in L_{2_e}(\mathbf{X})$, $e^{\sigma t}u(t) \in L_{2_e}(\mathbf{R}^m)$, and there exist η_1 and η_2 such that

$$\|e^{\sigma s}u(s)\|_t \leq \eta_1 + \eta_2\,\|e^{\sigma s}x(s)\|_t \quad .$$

■

Appendix XI Proof of Lemma 6

The proof basically follows the proof in Lemma 4.16 of [47] with slight modifications. Since $(A, Q^{\frac{1}{2}})$ is detectable, there exists $S \in \mathcal{L}(\mathbf{X})$ such that $A+SQ^{\frac{1}{2}}$ generates an exponentially stable C_o -semigroup $U_S(t)$. Write

$$A + BG = A + SQ^{\frac{1}{2}} + (BG - SQ^{\frac{1}{2}}) \quad .$$

Since $(BG-SQ^{\frac{1}{2}})$ is bounded, we can use the perturbation formula (3.1.2) in [1] to relate the C_o -semigroup $U_G(t)$ generated by $A+BG$ to the C_o -semigroup $U_S(t)$ generated by $A+SQ^{\frac{1}{2}}$:

$$U_G(t)x = U_S(t)x + \int_0^t U_S(t-\tau)(BG - SQ^{\frac{1}{2}})U_G(\tau)x\,d\tau \quad . \tag{A.11.1}$$

From (2.6),

$$Px = \int_0^\infty U_G^*(\tau)(Q + G^*RG)U_G(\tau)x\,d\tau \quad .$$

Since P is a bounded operator,

$$< Px, x >= \int_0^\infty \|Q^{\frac{1}{2}}U_G(\tau)x\|^2\,d\tau + \int_0^\infty \|R^{\frac{1}{2}}GU_G(\tau)x\|^2\,d\tau \leq \|P\|\,\|x\|^2 < \infty \quad .$$

This implies $Q^{\frac{1}{2}}U_G(\cdot)x \in L_2(\mathbf{R}_+;\mathbf{X})$. Since R is assumed to be coercive, $GU_G(\cdot)x \in L_2(\mathbf{R}_+;\mathbf{R}^m)$. From (A.11.1) ,

$$\|U_G(t)x\| \leq \|U_S(t)x\| + \int_0^t \left(\|U_S(t-\tau)\|^{\frac{1}{2}}\right)^2 (\|B\|\,\|GU_G(\tau)x\| + \|S\|\,\|Q^{\frac{1}{2}}U_G(\tau)x\|)\,d\tau \quad .$$

By Schwarz inequality,

$$\|U_G(t)x\| \leq \|U_S(t)x\| +$$
$$\left[\int_0^t \|U_S(t-\tau)\|\,d\tau\right]^{\frac{1}{2}} \times \left[\int_0^t \|U_S(t-\tau)\|(\|B\|\,\|GU_G(\tau)x\| + \|S\|\,\|Q^{\frac{1}{2}}U_G(\tau)x\|)^2 d\tau\right]^{\frac{1}{2}} \quad .$$

Squaring both sides and overbound square of sum by two times the sum of squares,

$$\|U_G(t)x\|^2 \leq 2\|U_S(t)x\|^2 + \\ c\int_0^t \|U_S(t-\tau)\|\,d\tau \times \int_0^t \|U_S(t-\tau)\|(\|GU_G(\tau)x\|^2 + \|Q^{\frac{1}{2}}U_G(\tau)x\|^2)\,d\tau \quad . \tag{A.11.2}$$

Since $U_S(t)$ is exponentially stable, there exists $M \geq 1$ and $\alpha > 0$ such that $\|U_S(t)\| \leq Me^{-\alpha t}$. The last term on the right hand side of (A.11.2) can be overbounded by

$$c_1\int_0^t e^{-\alpha(t-\tau)}(\|GU_G(\tau)x\|^2 + \|Q^{\frac{1}{2}}U_G(\tau)x\|^2)\,d\tau \quad .$$

Integrating this term with respect to t from 0 to ∞, we have

$$\begin{aligned}
&c_1\int_0^\infty\int_0^t e^{-\alpha(t-\tau)}(\|GU_G(\tau)x\|^2 + \|Q^{\frac{1}{2}}U_G(\tau)x\|^2)\,d\tau\,dt \\
&= c_1\int_0^\infty\int_\tau^\infty \left(e^{-\alpha(t-\tau)}\,dt\right)\left(\|GU_G(\tau)x\|^2 + \|Q^{\frac{1}{2}}U_G(\tau)x\|^2\right)d\tau \qquad (\text{ by Fubini Theorem }) \\
&= c_2\int_0^\infty (\|GU_G(\tau)x\|^2 + \|Q^{\frac{1}{2}}U_G(\tau)x\|^2)d\tau < \infty
\end{aligned}$$

The last inequality follows from the L_2-boundedness of GU_Gx and $Q^{\frac{1}{2}}U_Gx$ that have been shown earlier. Now, in (A.11.2) , $U_S(t)x$ is square integrable by using Datko Theorem and the exponential stability of $U_S(t)$, and we have shown that the second term is integrable. Hence, U_Gx is square integrable for all x, which, by Datko Theorem, implies that U_G is exponentially stable.

■

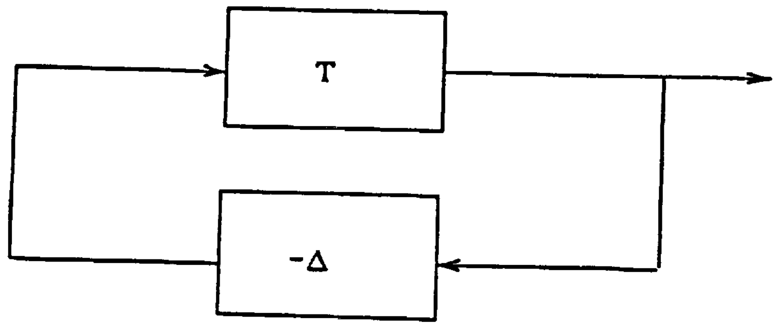


Fig. 1 Prototype Interconnected System

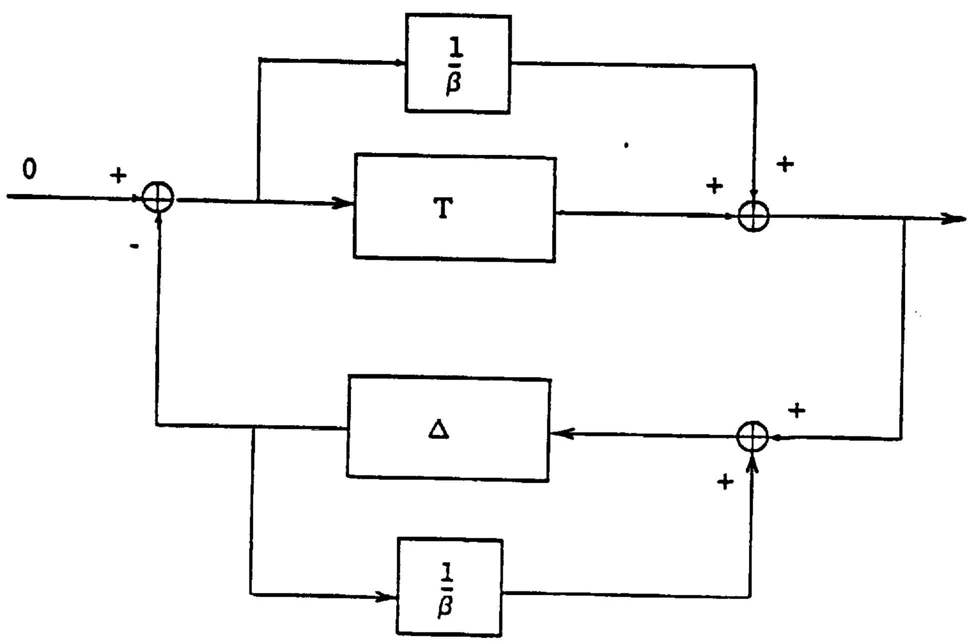


Fig.2 Interconnected System with Loop Transformation

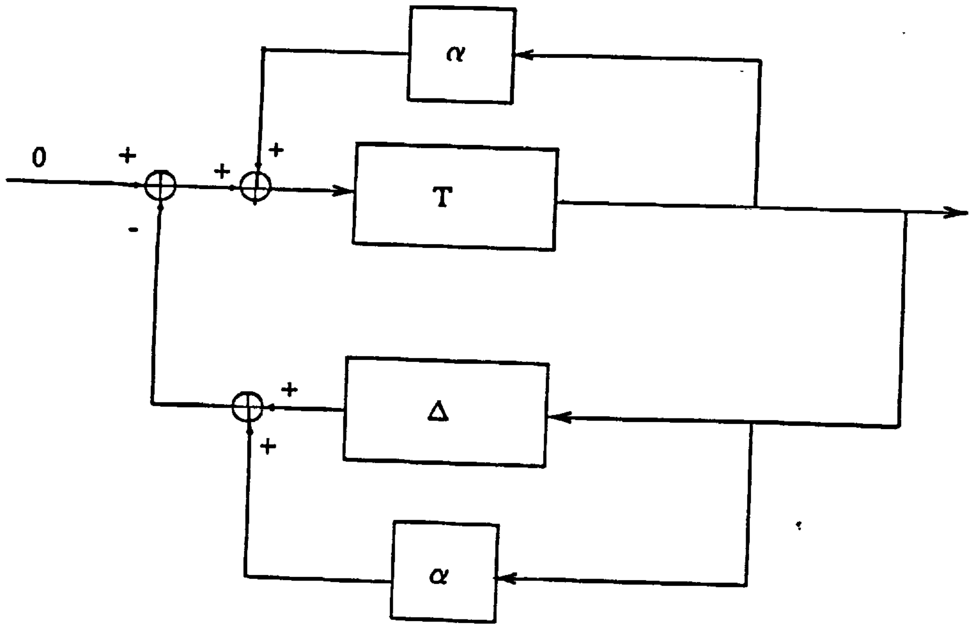


Fig.3 Interconnedted System with Loop Transformation

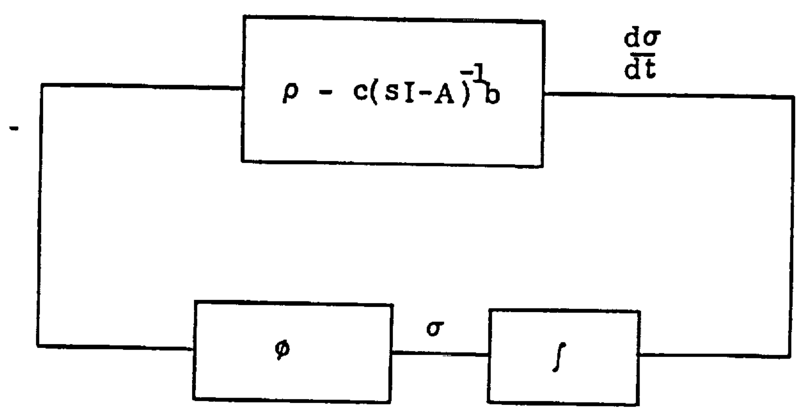


Fig.4 Interconnected System considered in [24]

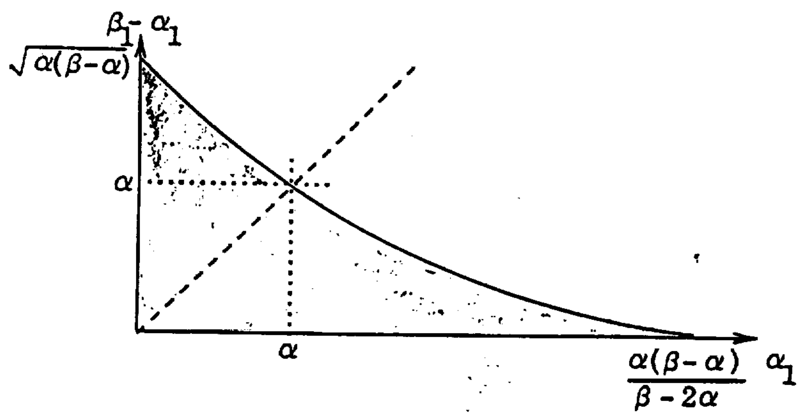


Fig. 5 Allowable (α_1, β_1) for given (α, β)

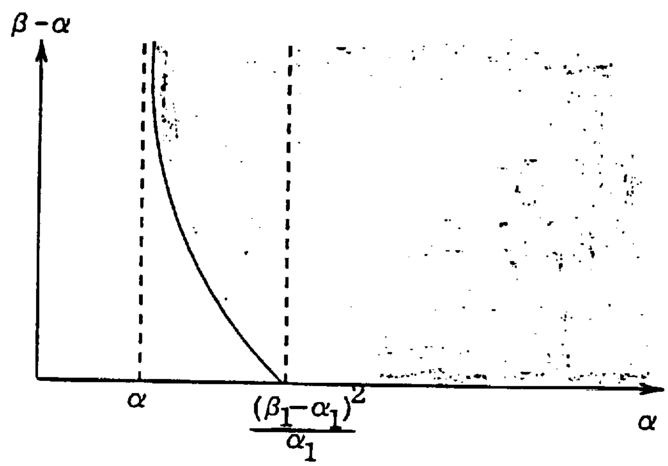


Fig. 6 Allowable (α, β) for given (α_1, β_1)

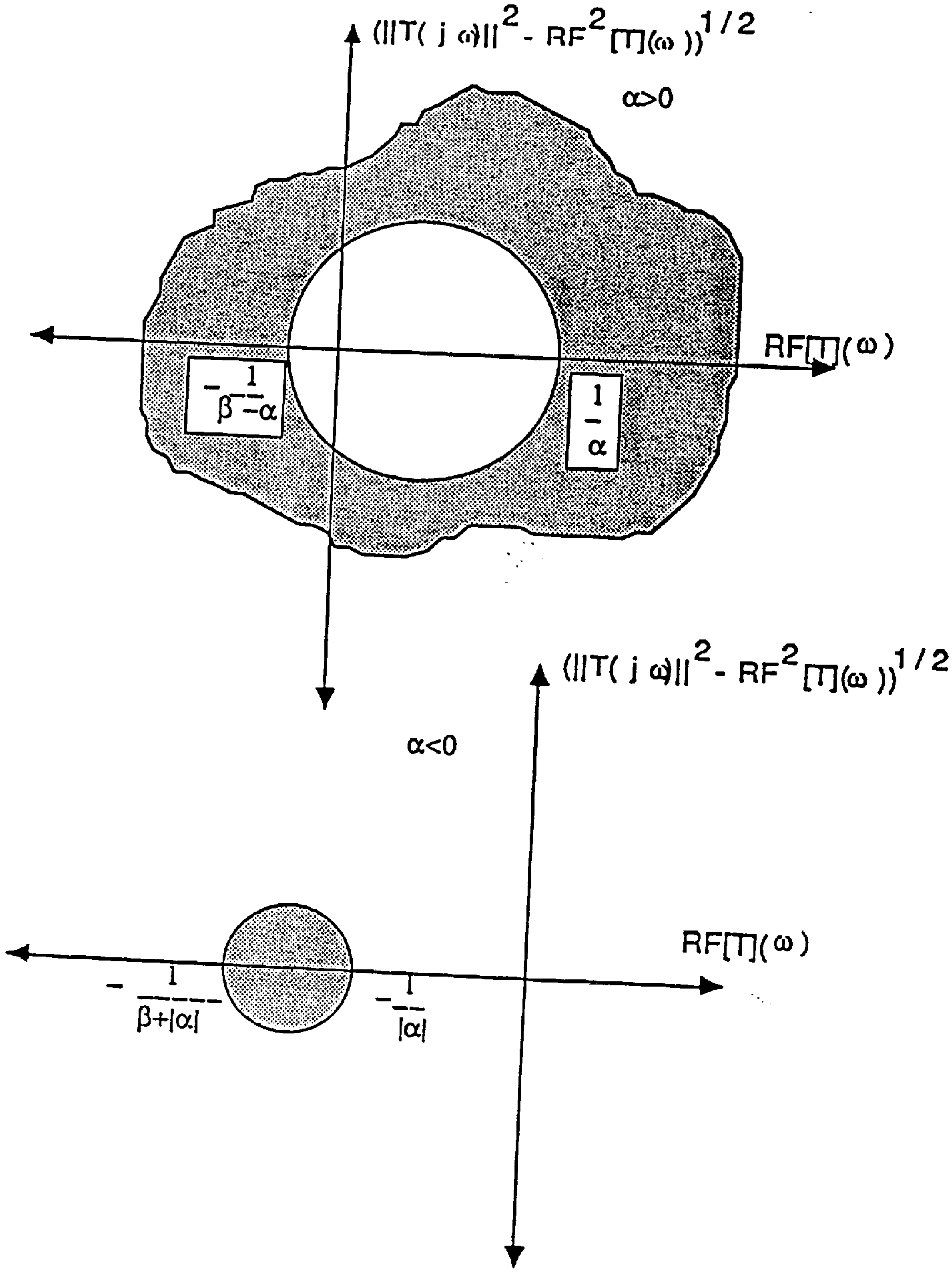


Fig. 7 Circle Test for Stability

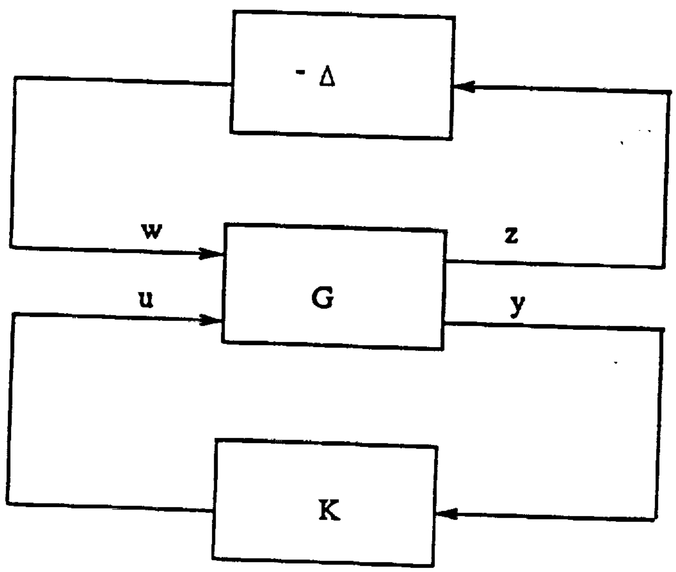


Fig. 8 Feedback Control System with Perturbation

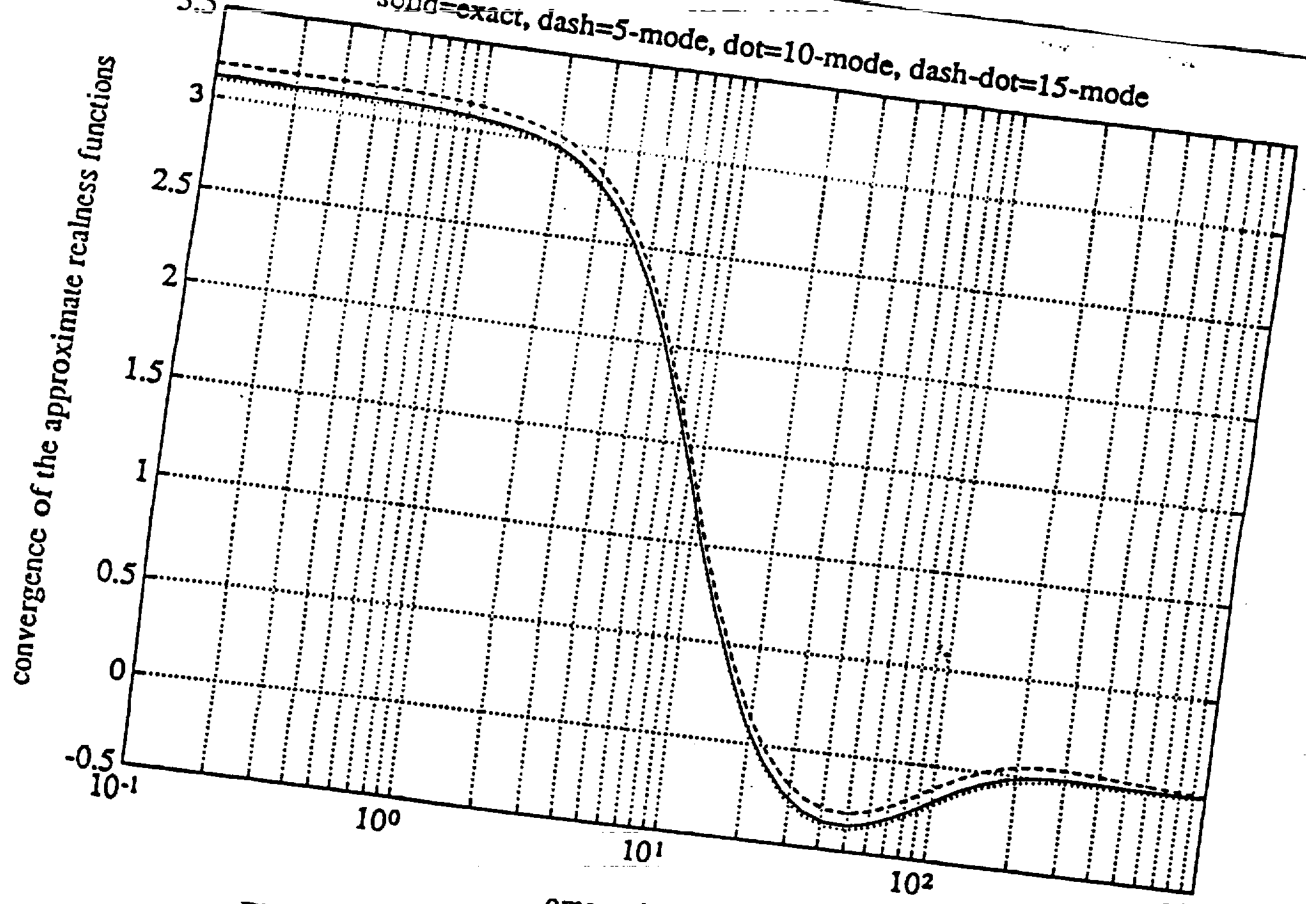


Fig. 9 Approximate and Exact Realness Functions for Example 3

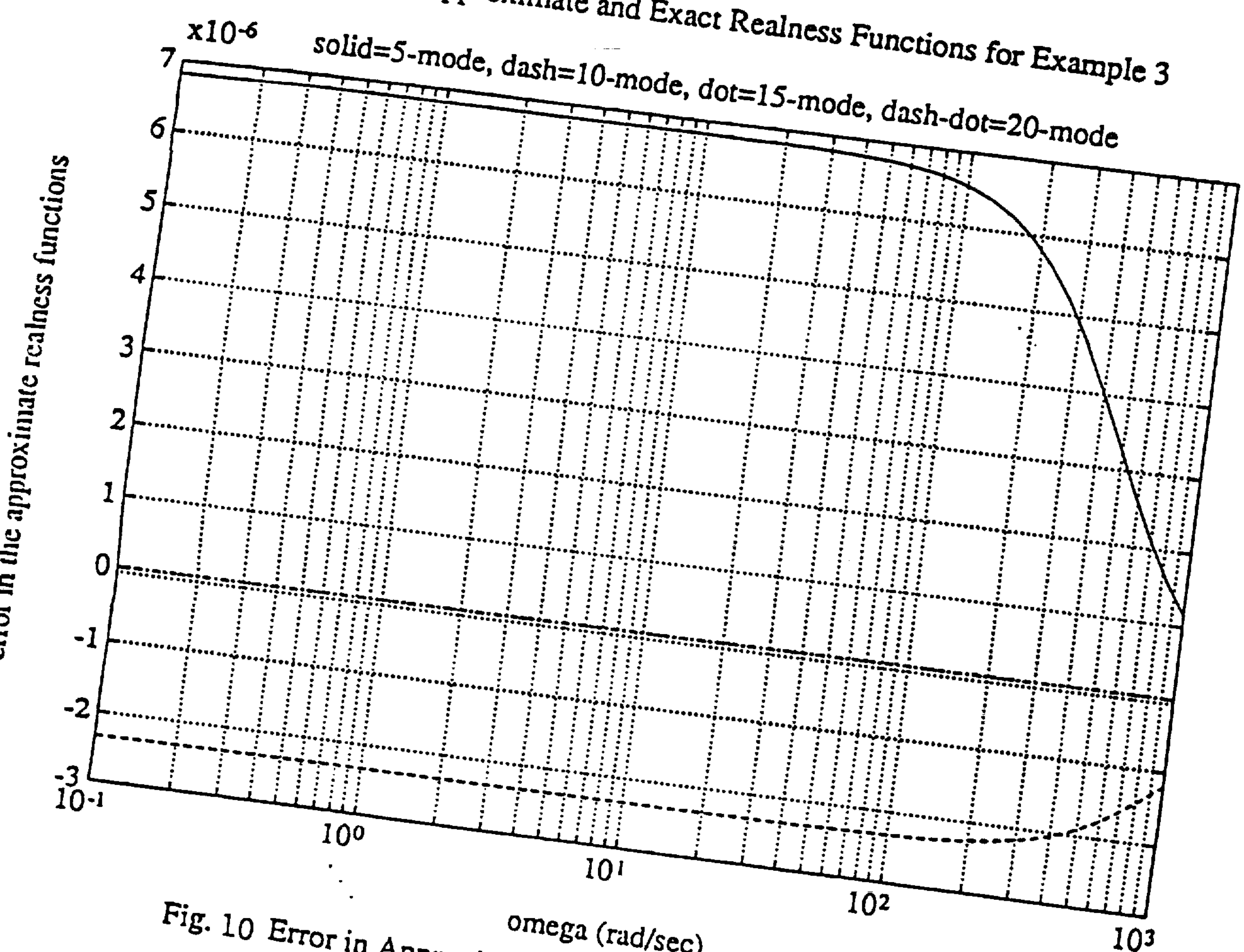
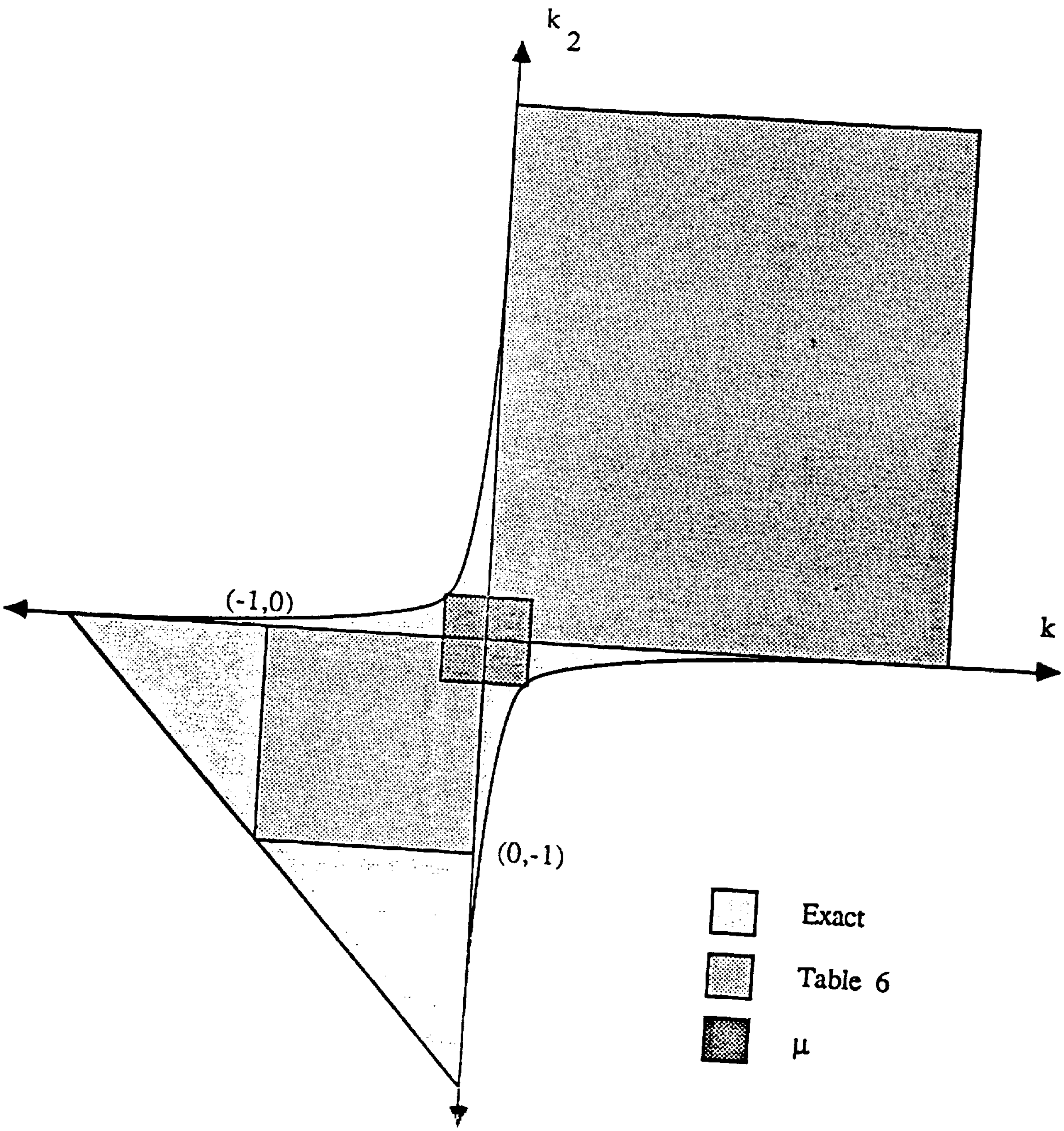


Fig. 10 Error in Approximate Realness Functions for Example 3



. Fig. 11 Stability Margin for Example 4

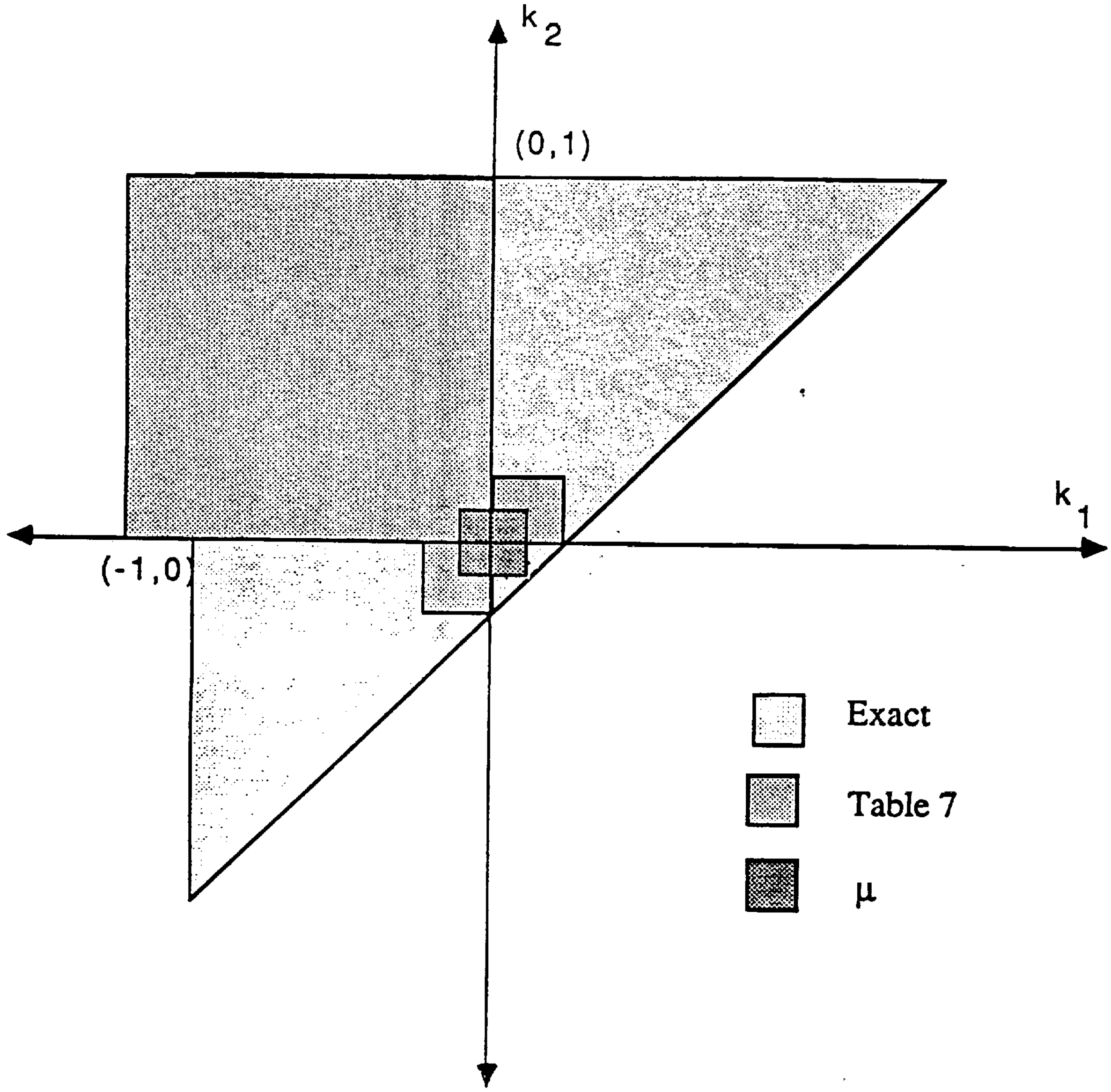


Fig. 12 Stability Margin for Example 5